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NOTES

ON THE

GEOMETRY OF THE PLANE TRIANGLE.

BY

JOHN GRIFFITHS, M.A.,
MATHEMATICAL LECTURER OF JESUS COLLEGE, OXFORD.

Oxford and London:
JAMES PARKER AND CO.
1867.

PRINTED · BY · J. PARKER & CO.
OXFORD

PREFACE.

SOME of the following papers have already appeared in various Mathematical Journals. These are now collected, and others are added. They are published, not as a complete treatise on the Geometry of the Triangle, but in the hope that they may be useful by way of reference.

The method adopted is the Trilinear, though in some cases Cartesian co-ordinates have been employed. The author would be glad if this slight sketch should induce some competent Mathematician to prosecute the subject further, and to write an exhaustive treatise on the properties of the Triangle.

JESUS COLLEGE, OXFORD,
June 1, 1867.

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NOTATION.

G, O, P are taken in the following pages to represent the centroid (or centre of gravity of the area), the centre of the circumscribed circle, and the point of intersection of the three perpendiculars, respectively, of any given triangle ABC.

A_1, B_1, C_1 are the middle points of the sides BC, CA, AB.

D, E, F the feet of the perpendiculars AP, BP, CP.

R, r, ρ the radii of the circumscribed, inscribed, and polar circles respectively.

Also (xyz) represents the circle which passes through the three points x, y, z; and (xy) the circle which has for a diameter the line joining x to y.

NOTES ON THE GEOMETRY OF THE PLANE TRIANGLE.

I.

Let x be any point on one of the sides—$B\,C$, say, of a given triangle $A\,B\,C$: then the polar circles of the triangles $A\,B\,x$, $A\,C\,x$, will pass through two fixed points, which lie on the polar circle of the triangle $A\,B\,C$.

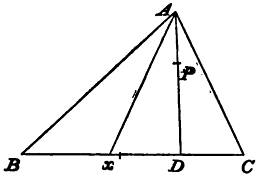

Fig. 1.

That these three circles are co-axal is easily inferred, from the fact that the polar circles of the four triangles that can be formed from any four lines are co-axal.

Also, since three of the sides of the quadrilateral before us—see fig. (1)—meet in A, it follows that this must be one of the limiting points of the system; and, therefore, D, the projection of A upon the side $B\,C$, must be the other; for given one limiting point, the other is the point of intersection of the line of centres with the polar of the first with respect to any circle of the system.

Hence, whatever the position of the point x may be on the line $B\,C$, the polar circles of the triangles $A\,B\,x$, $A\,C\,x$, will intersect that of the triangle $A\,B\,C$ in two fixed points, which lie on the line drawn perpendicular to the segment $A\,D$ through its middle point.

Consequences.

1°. It follows from the above that the polar circle of each of the triangles $A\,B\,x$, $A\,C\,x$, $A\,B\,C$, is cut orthogonally by the circle described upon the segment Ax as diameter; for this last circle passes through the limiting points A and D of the above system.

2°. If m denote the middle point of the segment Ax, the polar circle of the triangle ABC is the locus of the points common to the circles whose diameters are the lines Ax and Pm; for when two circles cut each other orthogonally, the circle which has for a diameter the line joining their centres passes through their two points of intersection.

3°. Hence, also, we may derive a well-known theorem, viz., that the polar circle is orthogonal to each of the circles which have the sides of the triangle for diameters.

It may be observed that since the limiting points A and D are real, the polar circles of the above triangles—ABx, ACx, ABC, will always intersect in imaginary points.

Also these three circles will be all real if either the angle B or C is an obtuse angle, and if x lie between B and C.

II.

Let x denote any point on the axis of homology of the triangles ABC, DEF—then the polar circle of the first triangle cuts orthogonally the circle described upon the segment Gx as diameter.

If ABC be taken as the triangle of reference, the trilinear equation to the axis of homology in question may be easily proved to be

$$\cos A\alpha + \cos B\beta + \cos C\gamma = 0, \tag{1}$$

and that to the polar circle to be

$$a\cos A\alpha^2 + b\cos B\beta^2 + c\cos C\gamma^2 = 0: \tag{2}$$

so that the pole of (1) with respect to (2) is the point $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, i.e. the centroid G.

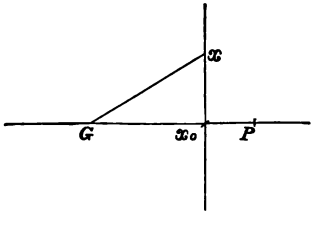

Fig. 2.

Hence, if x_0 be the point of intersection of (1) with the line GP, the point G is the inverse of x_0 relative to the polar circle (2): and, therefore, (see Townsend's "Modern Geometry," vol. i. art. 156,) this circle is orthogonal to any circle which passes through G and x_0.

Hence, also, if m be the middle point of the segment Gx, the polar circle is the locus of the points common to the circles whose diameters are the lines Gx and Pm.

III.

The polar circle of any triangle ABC is the locus of the point of intersection of rectangular tangents to each of the three hyperbolas which touch, respectively, the sides BC, CA, AB at their middle points A_1, B_1, C_1; and whose asymptotes are the pairs of lines PB, PC: PC, PA: PA, PB.

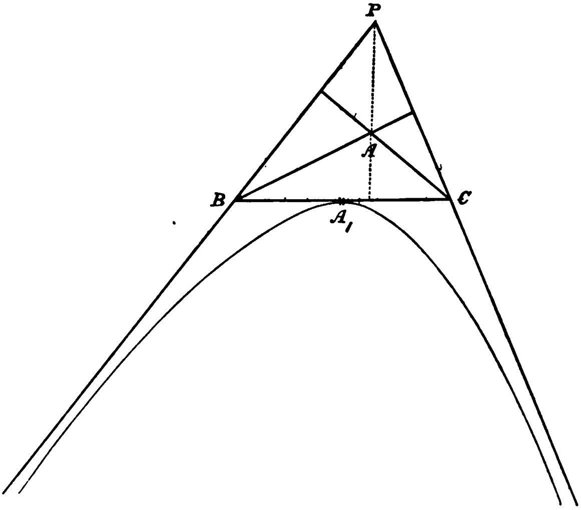

Fig. 3.

If we take the triangle *PB C*—see fig. (3)—as the triangle of reference, the equation to the hyperbola which touches *B C* at A_1, and whose asymptotes are *PB*, *P C* will be

$$\sqrt{-a_1\alpha}+\sqrt{b_1\beta}+\sqrt{c_1\gamma}=0,$$

where a_1, b_1, c_1 are the lengths of the sides *B C*, *CP*, *PB*.

Hence the equation to its director circle is

$$-a_1\cos A_1\alpha^2+b_1\cos B_1\beta^2+c_1\cos C_1\gamma^2-2c_1\cos A_1\gamma\alpha-2b_1\cos A_1\alpha\beta=0;$$

A_1, B_1, C_1 denoting the angles of the triangle of reference *P B C*.

(This may be easily deduced from the general expression given in Mr. Ferrers' "Trilinear Co-ordinates," p. 87.)

Now if we differentiate the above equation with respect to α, β, γ, we shall have the following:

$$a_1\alpha+b_1\beta+c_1\gamma=0,$$
$$\cos A_1\alpha-\cos B_1\beta=0,$$
$$\cos A_1\alpha-\cos C_1\gamma=0;$$

i.e. the centre of the director coincides with the point *P*; and the sides *CA*, *BA* are the polars with respect to it of the points *B* and *C*, respectively; or, the locus in question is the polar circle of the triangle *AB C*.

IV.

If a hyperbola, concentric with the polar circle, be supposed to pass through one of the vertices—*A*, say—of a given triangle *AB C*, so that its asymptotes are parallel to the adjacent sides *AB*, *A C*; then any chord of this curve which subtends a right angle at the polar centre *P* will be a tangent to the polar circle.

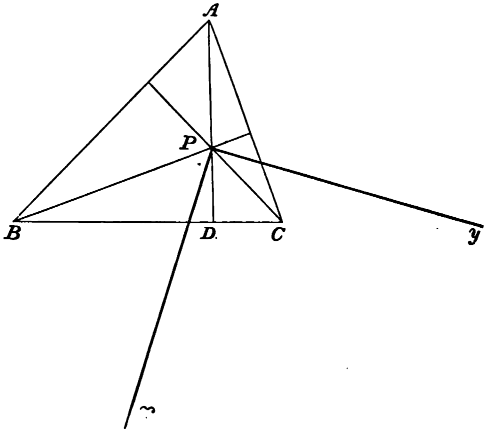

Fig. 4.

Let P be the origin, and Px, Py—the internal and external bisectors of the angle BPC—be the axes, of co-ordinates.

Then, if R = radius of circumscribed circle,

$$\angle BPx = 90^0 - \frac{A}{2} = CPx; \quad \text{see fig. (4.)}$$

$$PA = 2R \cos A,$$

$$\angle DPx = \frac{C-B}{2};$$

and the equation to the hyperbola in question will be

$$x^2 - \cot^2\frac{A}{2}\, y^2 = 4R^2 \cos^2 A\left(\cos^2\frac{B-C}{2} - \cot^2\frac{A}{2}\sin^2\frac{B-C}{2}\right),$$

or $$\sin^2\frac{A}{2}\, x^2 - \cos^2\frac{A}{2}\, y^2 = 4R^2 \cos^2 A \cos B \cos C.$$

Also, if $x \cos a + y \sin a - p = 0$, represent any chord of this curve, the equation of the lines joining its extremities to the origin is

$$p^2\left(\sin^2\frac{A}{2}\,x^2 - \cos^2\frac{A}{2}\,y^2\right) = 4R^2\cos^2A\cos B\cos C\,(x\cos\alpha + y\sin\alpha)^2;$$

and the condition that these shall be at right angles is easily seen to be

$$\left(p^2\sin^2\frac{A}{2} - 4R^2\cos^2A\cos B\cos C\cos^2\alpha\right)$$
$$-\left(p^2\cos^2\frac{A}{2} + 4R^2\cos^2A\cos B\cos C\sin^2\alpha\right) = 0,$$

or $$p^2 = -4R^2\cos A\cos B\cos C,$$

i.e. the chord which subtends a right angle at P is a tangent to the polar circle, $x^2 + y^2 = -4R^2\cos A\cos B\cos C$.

V.

The polar circle of any triangle ABC is orthogonal to the polar circle of each of the triangles PBC, PCA, PAB.

For the centre of the polar circle of the triangle PBC, for instance, is A; and its radius $= \sqrt{AP\,.\,AD}$

$$= \sqrt{2R\cos A\,.\,2R\sin B\sin C}$$
$$= 2R\sqrt{\cos A\sin B\sin C}.$$

Also the radius of the polar circle of the triangle ABC is

$$\sqrt{PA\,.\,PD},\ \text{or}\ 2R\sqrt{-\cos A\cos B\cos C}.$$

The condition that the two circles shall be orthogonal to each other is

$$(PA)^2 = 4R^2\cos A\sin B\sin C - 4R^2\cos A\cos B\cos C,$$
$$= 4R^2\cos^2A,$$

or $PA = 2R\cos A$,

an identity.

The polar circle is also orthogonal to the director of each of the four circles which touch the sides.

Or, generally, it is orthogonal to the director of any conic which touches the sides.

This theorem, which is the converse of the one referred to by Dr. Salmon in his "Conic Sections," p. 327, Ex. 2, may be proved directly as follows:—

Let $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$, be the equation of any conic touching α, β, γ; then the locus of a point such that the tangents drawn through it to this curve are at right angles to each other will be represented by

$$\begin{aligned} l\cos A\alpha^2 + m\cos B\beta^2 + n\cos C\gamma^2 - (l + m\cos C + n\cos B)\beta\gamma \\ - (m + n\cos A + l\cos C)\gamma\alpha \\ - (n + l\cos B + m\cos A)\alpha\beta = 0; \end{aligned}$$

which may be put under the form

$$lS_a + mS_b + nS_c = 0;$$

where

$$\begin{aligned} S_a &= \cos A\alpha^2 - \beta\gamma - \cos C\gamma\alpha - \cos B\alpha\beta, \\ S_b &= \cos B\beta^2 - \cos C\beta\gamma - \gamma\alpha - \cos A\alpha\beta, \\ S_c &= \cos C\gamma^2 - \cos B\beta\gamma - \cos A\gamma\alpha - \alpha\beta. \end{aligned}$$

Now the expressions S_a, S_b, S_c, when equated to zero, represent the circles described upon the sides of the triangle of reference as diameters; also any circle which is orthogonal to S_a, S_b, S_c must be also orthogonal to $lS_a + mS_b + nS_c$.

For if we form the Jacobian of the first three we have

$$J = \begin{vmatrix} \frac{dS_a}{d\alpha} & \frac{dS_a}{d\beta} & \frac{dS_a}{d\gamma} \\ \frac{dS_b}{d\alpha} & \frac{dS_b}{d\beta} & \frac{dS_b}{d\gamma} \\ \frac{dS_c}{d\alpha} & \frac{dS_c}{d\beta} & \frac{dS_c}{d\gamma} \end{vmatrix} \quad . \quad . \; 1^{\circ};$$

which, by the theory of determinants, is equivalent to any of the following forms:—

$$2^0.\ \begin{vmatrix} \frac{dS_a}{d\alpha} & \frac{dS_a}{d\beta} & \frac{dS_a}{d\gamma} \\ \frac{dS_b}{d\alpha} & \frac{dS_b}{d\beta} & \frac{dS_b}{d\gamma} \\ \frac{dS}{d\alpha} & \frac{dS}{d\beta} & \frac{dS}{d\gamma} \end{vmatrix};\quad 3^0.\ \begin{vmatrix} \frac{dS_a}{d\alpha} & \frac{dS_a}{d\beta} & \frac{dS_a}{d\gamma} \\ \frac{dS}{d\alpha} & \frac{dS}{d\beta} & \frac{dS}{d\gamma} \\ \frac{dS_c}{d\alpha} & \frac{dS_c}{d\beta} & \frac{dS_c}{d\gamma} \end{vmatrix};$$

$$4^0.\ \begin{vmatrix} \frac{dS}{d\alpha} & \frac{dS}{d\beta} & \frac{dS}{d\gamma} \\ \frac{dS_b}{d\alpha} & \frac{dS_b}{d\beta} & \frac{dS_b}{d\gamma} \\ \frac{dS_c}{d\alpha} & \frac{dS_c}{d\beta} & \frac{dS_c}{d\gamma} \end{vmatrix}:$$

where $S = lS_a + mS_b + nS_c$.

Hence, if any one of these determinants vanishes they will all do so; i.e. the circle which is orthogonal to any three of the circles in question is also orthogonal to the fourth.

Now it is proved in Sect. 1 that the polar circle is the orthotomic circle of S_a, S_b, S_c. Hence it is likewise orthogonal to $lS_a + mS_b + nS_c$; i.e. to the director of any conic touching α, β, γ.

Consequences.—α^0. The polar circle passes through the centre of any equilateral hyperbola touching the sides.

β^0. The directrix of any inscribed parabola is a diameter of the polar circle—a well-known theorem.

γ^0. We may also derive from the above the expressions for the distances of the polar centre from the centre of the inscribed and escribed circles, viz.,

$$PI = \sqrt{\rho^2 + 2r^2};\ PE_a = \sqrt{\rho^2 + 2r_a^2};\ \&c.$$

where I, E_a . . . denote the centres; and r, r_a . . . the radii of the circles which touch the sides.

VI.

If we suppose a conic to circumscribe ABC, so that the tangents drawn to it through the polar centre, P, are at right angles to each other: then the chord of contact of these tangents touches the polar circle.

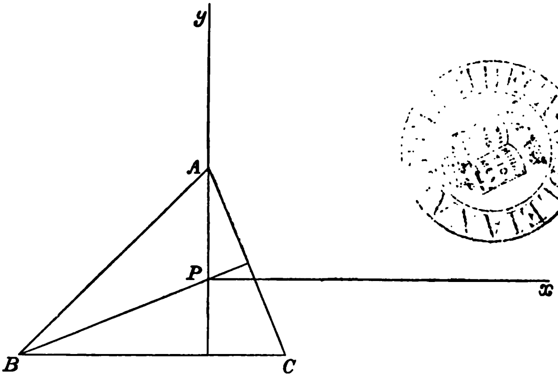

Fig. 5.

If the perpendicular PA be taken as the y-axis, and a line at right angles to it as the x-axis, the equations of the lines AB, AC will be easily found to be

$$-x\sin B+y\cos B=2R\cos A\cos B,$$

$$x\sin C+y\cos C=2R\cos A\cos C:$$

hence any conic passing through A, B, C may be represented by

$$\begin{gathered}l\,(-x\sin B+y\cos B-2R\cos A\cos B)\\(x\sin C+y\cos C-2R\cos A\cos C)\\+(y+2R\cos B\cos C)\{m(-x\sin B+y\cos B-2R\cos A\cos B)\\+n\,(x\sin C+y\cos C-2R\cos A\cos C)\}=0\,;\end{gathered}$$

which is equivalent to

$$\begin{gathered}-l\sin B\sin Cx^2+(l\cos B\cos C+m\cos B+n\cos C)\,y^2+2hxy\\+2gx+2fy-4R^2\cos A\cos B\cos C(-l\cos A+m\cos B+n\cos C)=0\,;\end{gathered}$$

$2h, 2g, 2f$ being written, for shortness, as the co-efficients of xy, x and y.

Now the two tangents through the origin to the curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

are represented by the equation

$$c(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$
$$= (gx + fy + c)^2;$$

and the condition that they shall be at right angles to each other is $c(a + b) = g^2 + f^2$.

Also the polar of the origin $gx + fy + c$ will touch the circle $x^2 + y^2 - \rho^2$, provided that $g^2 + f^2 = \frac{c^2}{\rho^2}$;

Hence, combining these two conditions, we find that if the polar in question subtends a right angle at the origin, it will touch the circle $x^2 + y^2 - \rho^2$, when $c = \rho^2 (a + b)$.

Now in the case of the conic before us this condition is satisfied, whatever the values of l, m, n may be, if

$$\rho^2 = -4R^2 \cos A \cos B \cos C.$$

For $c = -4R^2 \cos A \cos B \cos C\,(-l \cos A + m \cos B + n \cos C)$;

$$a + b = l \cos (B + C) + m \cos B + n \cos C$$
$$= -l \cos A + m \cos B + n \cos C,$$

or $$c = -4R^2 \cos A \cos B \cos C\,(a + b).$$

which proves the theorem.

The above admits of the following extension :—

Let K denote any conic circumscribing a given triangle ABC, such that the tangents to it from a given point P are harmonically conjugate with the lines joining P to two other fixed points Q, R. Then the envelope of the chord of contact of these tangents is a conic passing through Q, R; and self-conjugate with respect to the given triangle ABC.

VII.

The polar circles of the four triangles that can be formed from the sides BC, CA, AB, and either of the two tangents through the centre of the circumscribing circle to the maximum inscribed ellipse, will touch each other at a common point.

This theorem will be proved if we can shew that only one rectangular hyperbola can be drawn to touch the quadrilateral formed by the sides and one of the tangents in question.

Now the two equilateral hyperbolas which touch α, β, γ and $\lambda\alpha + \mu\beta + \nu\gamma$, are represented by the equation

$$\sqrt{L\alpha} + \sqrt{M\beta} + \sqrt{N\gamma} = 0:$$

where

$$L^2 + M^2 + N^2 + 2MN\cos A + 2NL\cos B + 2LM\cos C = 0,$$

and $$\frac{L}{\lambda} + \frac{M}{\mu} + \frac{N}{\nu} = 0;$$

also the condition that these curves shall coincide is

$$\frac{\sin^2 A}{\lambda^2} + \frac{\sin^2 B}{\mu^2} + \frac{\sin^2 C}{\nu^2} + 2\,\frac{(\cos B\cos C - \cos A)}{\mu\nu}$$
$$+ 2\,\frac{(\cos C\cos A - \cos B)}{\nu\lambda}$$
$$+ 2\,\frac{(\cos A\cos B - \cos C)}{\lambda\mu} = 0,$$

or $(\sin A\mu\nu + \sin B\nu\lambda + \sin C\lambda\mu)^2 = 4\lambda\mu\nu(\lambda\cos A + \mu\cos B + \nu\cos C)$.

This equation, then, represents the envelope of a variable line (λ, μ, ν), such that only one rectangular hyperbola can be drawn to touch it and the sides of the triangle of reference.

Hence, since the equation is satisfied when $\lambda\cos A + \mu\cos B + \nu\cos C = 0$, and $\sin A\mu\nu + \sin B\nu\lambda + \sin C\lambda\mu = 0$, simultaneously, we see that the line (λ, μ, ν) will in two instances coincide with the tangents through the point $\lambda\cos A + \mu\cos B + \nu\cos C$, to the conic $\sin A\,\mu\nu + \sin B\,\nu\lambda + \sin C\lambda\mu$; i.e. with the tangents through the centre of the circumscribing circle to the maximum inscribed ellipse.

NOTE. The above envelope is evidently a curve of the fourth class, which passes through A, B, C and the centre of the circumscribed circle; and touches the sides BC, CA, AB.

If the triangle is acute-angled, O, the centre of circumscribed circle, will lie *inside* the maximum ellipse in question; and the two tangents will be imaginary.

If the triangle is right-angled, O will lie *on* the ellipse, and the system of polar circles will reduce to a point.

Lastly, if the triangle is obtuse-angled, O will lie *outside* the curve, and the two tangents, together with the polar circles, will be all real.

If $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ denote the points Q_1, Q_2 which are common to the polar and circumscribed circles, the equations of the two tangents through O to the maximum inscribed ellipse will be

$$\alpha_1\alpha + \beta_1\beta + \gamma_1\gamma = 0,$$

$$\alpha_2\alpha + \beta_2\beta + \gamma_2\gamma = 0;$$

for the line $l\alpha + m\beta + n\gamma = 0$, will pass through O, and touch the curve $\sqrt{a\alpha} + \sqrt{b\beta} + \sqrt{c\gamma} = 0$, provided that

$$l\cos A + m\cos B + n\cos C = 0,$$

$$\text{and}\quad \frac{a}{l} + \frac{b}{m} + \frac{c}{n} = 0;$$

i.e. provided that l, m, n are proportional to the co-ordinates of either of the points Q_1, Q_2.

Hence, we have the following construction for the two tangents in question:—

Let R_1, R_2 be the points inverse to Q_1, Q_2 with respect to the sides of the triangle of reference, then the required tangents are the polars relative to this triangle of R_1, R_2.

The expression Polar of a point, in regard to a triangle, is due to Professor Cayley, and denotes "a line constructed as follows:—viz. R being the point and ABC the triangle, then taking on BC a point a, the harmonic in regard to the points B and C of the intersection of BC by AR; and in like manner on CA and AB the points b and c respectively, the three points a, b, c lie on a line which is the polar of the point R. If the equations of the sides are $x = 0$, $y = 0$, $z = 0$, and the co-ordinates of the point are α, β, γ, then the equation of the polar is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0."$$

(See the "Educational Times," vol. i. p. 77.)

VIII.

If (xy) denote the circle described upon the line joining the points x and y as diameter; the six points of intersection of the three pairs of circles (AA_1), (BC); (BB_1), (CA); (CC_1), (AB) lie on the circumference of another circle, whose centre coincides with the centre of gravity G, and which cuts orthogonally the polar circle.

The equation to the circle (BC) must be of the form

$$k(a\alpha + b\beta + c\gamma)\alpha = a\beta\gamma + b\gamma\alpha + c\alpha\beta;$$

also, since the curve passes through E, $(\cos C, 0, \cos A)$,

$$k = \cos A,$$

$\therefore$ $(BC) = \cos A\alpha^2 - \beta\gamma - \cos C\gamma\alpha - \cos B\alpha\beta = 0$, represents the circle (BC).

In the same way it may be shewn that the equation to the circle (AA_1) is

$$(AA_1) = b\cos B\beta^2 + c\cos C\gamma^2 - a\beta\gamma - (b + c\cos A)\gamma\alpha \\ - (c + b\cos A)\alpha\beta = 0:$$

hence, it follows that

$$a.(BC) + (AA_1) = \Sigma a \cos A\alpha^2 - 2\Sigma a\beta\gamma = b.(CA) + (BB_1)$$
$$= c.(AB) + (CC_1);$$

relations which prove the first part of the theorem.

The equations to the radical axes of (BC), (AA_1); (CA), (BB_1); (AB), (CC_1) are

$$-2\cos A\alpha + \cos B\beta + \cos C\gamma = 0,$$
$$\cos A\alpha - 2\cos B\beta + \cos C\gamma = 0,$$
$$\cos A\alpha + \cos B\beta - 2\cos C\gamma = 0;$$

whence it follows that they intersect in the polar centre P.

The second part may be proved by shewing that the circle in question is the director of the ellipse which touches the sides of the triangle at their middle points. For it has been shewn in (V.) that the director of any inscribed conic is orthogonal to the polar circle.

The following is, however, a more direct proof.

I shall first find the centre and radius of the circle represented by the general equation

$$\Sigma a \cos A\alpha^2 + k\Sigma a\beta\gamma = 0;$$

k being an indeterminate constant.

If T, T' denote the lengths of the tangents from (α, β, γ) to the circles $\Sigma a \cos A\alpha^2$, and $\Sigma a\beta\gamma$ respectively, it may be shewn (see Dr. Salmon's "Conic Sections," 4th Edit., p. 125) that

$$(T)^2 = \frac{a\cos A\alpha^2 + b\cos B\beta^2 + c\cos C\gamma^2}{2R\sin A\sin B\sin C},$$

$$(T')^2 = \frac{a\beta\gamma + b\gamma\alpha + c\alpha\beta}{-2R\sin A\sin B\sin C}.$$

Hence the circle $\Sigma a\cos A\alpha^2 + k\Sigma a\beta\gamma$ is the locus of a point such that $\frac{(T)^2}{k} = \frac{(T')^2}{1}$; and, therefore, (see Townsend's "Modern

Geometry," vol. i. p. 257,) its centre is the point which divides the line OP in the ratio $1 : k$.

For instance, if $k = -1$, the locus will be a circle whose centre bisects the segment OP.

(When k is *negative*, the centre in question divides OP *internally*.)

If $k = -2$; the centre will be the point which divides OP internally in the $1 : 2$; i.e. the centroid G.

Again, the radius may be found immediately from the formula which connects the radii of any three co-axal circles ("Modern Geometry," vol. i. p. 255).

The expression for it will be found to be

$$\frac{k}{k-1}\sqrt{1 - \frac{4}{k^2}(1+k)\cos A \cos B \cos C} \,.\, R;$$

R being the radius of the circle (ABC).

$$\text{If } k = -1;\ \text{the radius} = \frac{R}{2}$$

$$\text{If } k = -2;\qquad ,, \qquad = \frac{2}{3}\sqrt{1 + \cos A \cos B \cos C} \,.\, R.$$

To prove, then, that the circle $\Sigma a \cos A\alpha^2 - 2\Sigma a\beta\gamma$ is orthogonal to the polar circle $\Sigma a \cos A\alpha^2$, we have simply to shew that

$$PG^2 = -4\cos A \cos B \cos C \,.\, R^2 + \frac{4}{9}(1 + \cos A \cos B \cos C) \,.\, R^2,$$

or

$$PG^2 = \frac{4}{9}(1 - 8\cos A \cos B \cos C) R^2,$$

which is known to be the case.

IX.

The six points common to the three pairs of circles (AA_1), (PBC); (BB_1), (PCA); (CC_1), (PAB) lie on the circumference of a circle, whose centre is the point which divides the segment OP internally in the ratio $2 : 3$.—(See explanation of the Notation.)

The equation to the circle (PBC) must be of the form

$$k\alpha(a\alpha + b\beta + c\gamma) = a\beta\gamma + b\gamma\alpha + c\alpha\beta,$$

where $$k = 2\cos A:$$

hence $$(PBC) = 2a\cos A\alpha^2 - a\beta\gamma - (a\cos C - c\cos A)\gamma\alpha$$
$$- (a\cos B - b\cos A)\alpha\beta,$$

and $$AA_1 = b\cos B\beta^2 + c\cos C\gamma^2 - a\beta\gamma - (b + c\cos A)\gamma\alpha$$
$$- (c + b\cos A)\alpha\beta,$$

$$\therefore 2(AA_1) + (PBC) = 2\Sigma a\cos A\alpha^2 - 3\Sigma a\beta\gamma$$
$$= 2(BB_1) + (PCA) = 2CC_1 + (PAB);$$

equalities which prove the theorem.

X.

The circles (ABC), $(A_1B_1C_1)$, and (PG) are co-axal.

It has been shewn in (VIII.) that the equation

$$\Sigma a\cos A\alpha^2 - \Sigma a\beta\gamma = 0,$$

represents a circle whose centre bisects the segment OP, and whose radius $= \frac{R}{2}$.

Now if $\alpha = 0$, this equation becomes

$$b\cos B\beta^2 + c\cos C\gamma^2 - a\beta\gamma = 0,$$

or $$(b\beta - c\gamma)(\cos B\beta - \cos C\gamma) = 0;$$

i.e. the circle represented by it passes through A_1 and D; and, similarly, through B_1, E and C_1, F.

Hence, the two triangles $A_1 B_1 C_1$ and DEF are inscribed in a circle whose centre bisects the segment OP, and whose radius $= \frac{R}{2}$.

Whence it follows that the points P and G are the external and internal centres of similitude of the circles (ABC), $(A_1 B_1 C_1)$; and, therefore, that the circle whose diameter is the line PG passes through their points of intersection.

The circle (PG) is, in fact, the locus of the points at which the circles (ABC), $(A_1 B_1 C_1)$ subtend equal angles.

For if T', T'' denote the lengths of the tangents from (α, β, γ) to these respective circles, we have

$$T' = \sqrt{\frac{a\beta\gamma + b\gamma\alpha + c\alpha\beta}{-2R\sin A \sin B \sin C}},$$

$$T'' = \sqrt{\frac{\Sigma a \cos A\alpha^2 - \Sigma a\beta\gamma}{4R\sin A \sin B \sin C}};$$

hence, putting $\frac{T'}{R} = \frac{T''}{\frac{R}{2}}$, the equation to the locus of the points at which the circles in question appear of equal size, will be

$$2(\Sigma a \cos A\alpha^2 - \Sigma a\beta\gamma) = -\Sigma a\beta\gamma,$$

$$\text{or} \quad 2\,\Sigma a \cos A\alpha^2 - \Sigma a\beta\gamma = 0,$$

which is the same as

$$(b\beta - c\gamma)(\cos B\beta - \cos C\gamma) + (c\gamma - a\alpha)(\cos C\gamma - \cos A\alpha) \\ + (a\alpha - b\beta)(\cos A\alpha - \cos B\beta) = 0:$$

i.e. the locus is the circle (PG).

Consequences.

If the above theorem be applied to the triangle $A_1 B_1 C_1$, we learn that the nine-point circle of ABC, the circle which bisects the "medianes" AA_1, BB_1, CC_1, and the circle whose diameter is the line GO are co-axal.

XI.

On the segments PA, PB, PC—(see Notation)—let three points p, q, r be taken such that $Pp = \frac{1}{n} . PA$; $Pq = \frac{1}{n} . PB$; $Pr = \frac{1}{n} . PC$; and on PA_1, PB_1, PC_1 three other points p', q', r', such that $Pp' = \frac{2}{n} . PA_1$; $Pq' = \frac{2}{n} . PB_1$; $Pr' = \frac{2}{n} . PC_1$.

Then 1°. the points p, q, r, p', q', r' lie on the circumference of a circle which has for a diameter any one of the lines pp', qq', rr'.

2°. The centre of this circle coincides with the point M, taken on the line PO such that $PM = \frac{1}{n} . PO$; and its radius $= \frac{1}{n} . R$.

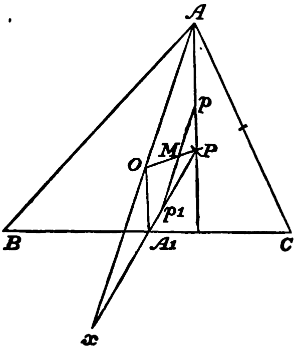

Fig. 6.

Produce AO, PA_1, (fig. 6) to meet in x: and join OA_1; then by similar triangles, we have

$$xO : xA = A_1O : PA = 1 : 2;$$

O is, therefore, the point of bisection of Ax, and A_1 that of Px.

Again, since $Pp : PA = Pp' : Px$ (by hypothesis), the line pp' is parallel to Ax, and must, therefore, cut PO in the point M such that $PM = \frac{1}{n} . PO$, and $pM = \frac{1}{n} . AO$.

Similarly, qq' and rr' may be shewn to intersect PO in M.

Hence, since $AO = BO = CO = R$, the six points p, p', q, q', r, r' will lie on a circle whose centre is M and radius $\frac{1}{n} . R$.

Consequences.

If $\frac{1}{n} = \frac{1}{2}$, we have the nine-point circle.

If $\frac{1}{n} = \frac{2}{3}$, it follows that the circle whose radius $= \frac{2}{3} R$, and whose centre coincides with the centroid of the triangle, cuts the segments PA, PB, PC internally in the ratio of $2 : 1$; and the segments PA_1, PB_1, PC_1, produced, in the ratio $4 : 1$.

I shall now proceed to find the condition that the circle $(pp' qq' rr')$ shall touch the circle inscribed in the triangle ABC.

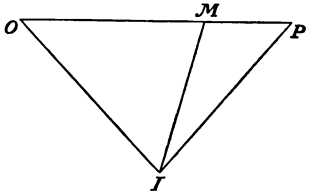

Fig. 7.

Let M and I denote the centres of the two circles in question; then if they touch we must have

$$MI = \frac{1}{n} . R \pm r.$$

Also—see fig. (7)—

$$\frac{PO^2 + PI^2 - OI^2}{2\,PO\,.\,PI} = \cos OPI = \frac{PM^2 + PI^2 - MI^2}{2\,PM\,.\,PI};$$

$$\left.\begin{aligned} PM &= \frac{1}{n}\,.\,PO \\ OI^2 &= R^2 - 2\,Rr \\ PI^2 &= 2\,r^2 + \rho^2 \\ PO^2 &= R^2 + 2\,\rho^2 \end{aligned}\right\}$$

Hence

$$(2\,r^2 + 3\rho^2 + 2\,Rr) = n\,.\left(\frac{R^2 + 2\rho^2}{n^2} + 2r^2 + \rho^2 - \left(\frac{R}{n} \pm r\right)^2\right),$$

or $$\frac{2\,r^2 + 3\,\rho^2 + 2\,Rr}{n} = \frac{2\,\rho^2}{n^2} + \rho^2 + r^2 \mp 2\,\frac{R}{n}r.$$

Supposing, then, that the + sign be taken, we have

$$\frac{2\rho^2}{n^2} + \rho^2 + r^2 = \frac{2\,r^2 + 3\,\rho^2}{n}.$$

1°. Let $\rho = 0$; i.e. let the triangle be right-angled, then $\frac{1}{n} = \frac{1}{2}$.

2°. Let $\rho \gtrless .\,0$; then $\frac{1}{n} = \frac{1}{2}$ or $1 + \frac{r^2}{\rho^2}$.

Hence in *any* triangle, the nine-point circle touches internally the inscribed circle.

In the same way, if we sought the condition that the circle (M) may touch any one of the escribed circles, we should find $\frac{1}{n} = \frac{1}{2}$; i.e. the nine-point circle touches the inscribed and escribed circles.

["Le théorème sur le contact du cercle des neuf points, attribué quelquefois à Terquem, d'autres fois à Steiner, est revendiqué par Steiner lui-même pour son compatriote Feuerbach." —*Nouvelles Annales de Mathématiques*, 1863, p. 562.]

Again, for the cases of external contact the values of n are given by the quadratic

$$\frac{2\rho^2}{n^2}+r^2+\rho^2=\frac{2r^2+3\rho^2+4Rr}{n}.$$

If $\rho=0$, $\frac{1}{n}=\frac{r}{2r+4R}$; i.e. if ABC be a triangle whose angle A —say—is a right angle, the circle which divides the sides AB, AC in the ratio $1:1+4\frac{R}{r}$, and the "mediane" AA_1 in the ratio $1:\frac{2R}{r}$ passes through A, and touches the inscribed circle.

The condition that the circle (M) shall be orthogonal to the polar circle is

$$(PM)^2=\frac{R^2}{n^2}+\rho^2$$

$$\frac{R^2+2\rho^2}{n^2}=\frac{R^2}{n^2}+\rho^2,$$

or $$n=\sqrt{2}.$$

Hence, if p, q, r denote the points on PA, PB, PC such that Pp, Pq, Pr are the sides of three squares, whose diagonals are the segments PA, PB, PC respectively, the circle $(p\,q\,r)$ will be orthogonal to the polar circle of the triangle ABC.

XII.

Through each angular point of any triangle ABC let a line be drawn parallel to the opposite side; and let A', B', C' be the vertices, opposite to A, B, C, of the triangle thus formed. Then the nine-point circle of the triangle ABC touches the nine-point circle of each of the triangles $A'BC$, $B'CA$, $C'AB$.

The truth of this theorem may be seen at once by an inspection of fig. (8)—where A_1—the middle point of BC—is the centre of similitude of the two triangles ABC, $A'BC$; P, P' the points of intersection of their perpendiculars; and m, m' the middle points of the segments PA, $P'A'$.

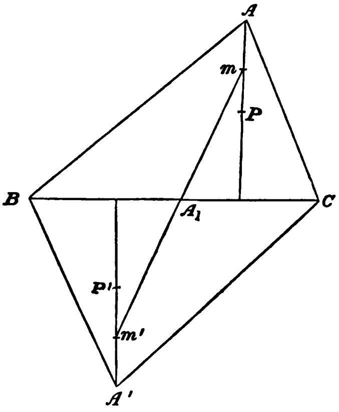

Fig. 8.

Consequences.

The centres of the circles which bisect the sides of the triangles $A'BC$, $B'CA$, $C'AB$ lie on the circumference of a circle whose centre coincides with that of the nine-point circle of the triangle ABC, and whose radius $= R$.

XIII.

The middle points of the nine segments AA_1, BB_1, CC_1, OA_1, OB_1, OC_1, OD, OE, OF—(see Notation)—lie on the circumference of a circle of radius $\frac{1}{4}R$.

For the circle which bisects the segments AA_1, BB_1, CC_1 bisects at the same time the sides B_1C_1, C_1A_1, A_1B_1.

Also, $-O$ is the polar centre of the triangle $A_1B_1C_1$; i.e. it is the external centre of similitude of its nine-point and circumscribing circles, whose ratio of similitude is $1:2$.

XIV.

The common nine-point circle of the four triangles *PBC*, *PCA*, *PAB*, *ABC* touches the two tangents drawn from any one of the four points *A*, *B*, *C*, *P* to the circle which passes through the other three.

This follows immediately from the fact that the polar centre of any triangle is a centre of similitude of the nine-point and circumscribing circles of that triangle; and also that any one of the four points in question is the polar centre of the triangle formed by joining the other three.

XV.

If a line be drawn from the polar centre, *P*, parallel to either of the radii *AO*, *BO*, *CO*—to *AO*, say, and if x_1 be the point where it meets the opposite side *BC*; then the nine-point circle touches each of the two circles which have for diameters the segments Px_1, Ox_1.

The reader may prove this for himself, by shewing that x_1 is the point where the inscribed conic, whose foci are *P* and *O*, meets the side *BC*: for the nine-point circle is easily shewn to be the auxiliary circle of the conic in question; and any circle described upon a focal radius vector as diameter touches the auxiliary circle.

XVI.

The nine-point circles of the triangles that can be formed from *A*, *B*, *C*, *P*, *G* together with the maximum ellipse that can be inscribed in the triangle *ABC*, all pass through a common point, viz. the centre of the equilateral hyperbola (*ABCPG*).

This theorem may be deduced from the following propositions.

1°. The ellipse in question is the locus of the centres of all the conics which circumscribe the quadrangle *A*, *B*, *C*, *G*.

2°. The nine-point circle of any triangle is the locus of the centres of equilateral hyperbolas described about that triangle.

XVII.

The feet of the perpendiculars let fall upon the sides of the triangle of reference from the two points whose co-ordinates are ξ, η, ζ and $\frac{1}{\xi}, \frac{1}{\eta}, \frac{1}{\zeta}$, lie on the circumference of the same circle.

The co-ordinates of the foot of the perpendicular from (ξ, η, ζ) upon the side BC are easily found to be

$$\alpha = 0, \frac{\beta}{\eta + \xi \cos C} = \frac{\gamma}{\zeta + \xi \cos B},$$

and those of the projection of $\left(\frac{1}{\xi}, \frac{1}{\eta}, \frac{1}{\zeta}\right)$ upon the same line to be

$$\alpha = 0, \frac{\beta}{\frac{1}{\eta} + \frac{1}{\xi}\cos C} = \frac{\gamma}{\frac{1}{\zeta} + \frac{1}{\xi}\cos B}.$$

The equations $\alpha = 0$, and

$$\{(\zeta+\xi\cos B)\beta-(\eta+\xi\cos C)\gamma\}\left\{\left(\frac{1}{\zeta}+\frac{1}{\xi}\cos B\right)\beta-\left(\frac{1}{\eta}+\frac{1}{\xi}\cos C\right)\gamma\right\}=0,$$

will, then, when taken simultaneously, give us the co-ordinates of the projections of (ξ, η, ζ), $\left(\frac{1}{\xi}, \frac{1}{\eta}, \frac{1}{\zeta}\right)$ upon the side BC.

Hence it is readily seen that the feet of the perpendiculars in question will lie on a conic whose equation is

$$\Sigma(\eta+\zeta\cos A)\left(\frac{1}{\eta}+\frac{1}{\zeta}\cos A\right)\alpha^2$$

$$-\Sigma\left\{(\eta+\xi\cos C)\left(\frac{1}{\zeta}+\frac{1}{\xi}\cos B\right)+\left(\frac{1}{\eta}+\frac{1}{\xi}\cos C\right)(\zeta+\xi\cos B)\right\}\beta\gamma=0,$$

or

$$\Sigma(1+x\cos A+\cos^2 A)\alpha^2-\Sigma(2\cos B\cos C+x+y\cos C+z\cos B)\beta\gamma=0, \tag{1}$$

where $\dfrac{\eta}{\zeta}+\dfrac{\zeta}{\eta}=x$; $\dfrac{\zeta}{\xi}+\dfrac{\xi}{\zeta}=y$; $\dfrac{\xi}{\eta}+\dfrac{\eta}{\xi}=z$.

If this equation be written

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0,$$

the conditions that it shall represent a circle are (see Mr. Ferrers' "Trilinear Co-ordinates," p. 85)

$$wb^2+vc^2-2u'bc=uc^2+wa^2-2v'ca=va^2+ub^2-2w'ab.$$

Now in the case before us

$$\begin{aligned}wb^2+vc^2-2u'bc&=(1+z\cos C+\cos^2C)\,b^2+(1+y\cos B+\cos^2B)c^2\\&\quad+(2\cos B\cos C+x+y\cos C+z\cos B)\,bc\\&=a^2+b^2+c^2+abc\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\\&=uc^2+wa^2-2v'ca=va^2+ub^2-2w'ab.\end{aligned}$$

Or the curve is a circle.

If (1) be multiplied by $\sin A\sin B\sin C$, $\xi\eta\zeta$, it becomes

$$\begin{aligned}&(\sin A\alpha+\sin B\beta+\sin C\gamma)\,\Sigma\xi(\eta+\zeta\cos A)(\eta\cos A+\zeta)\sin B\sin C\alpha\\&=(\sin A\xi+\sin B\eta+\sin C\zeta)(\sin A\eta\zeta+\sin A\zeta\xi+\sin C\xi\eta)\,\Sigma\sin A\beta\gamma.\end{aligned}\tag{2}$$

If the circle in question be called the pedal circle of (ξ,η,ζ), it will be seen that the nine-point circle, for instance, is the pedal of either $(\cos A, \cos B, \cos C)$ or $(\sec A, \sec B, \sec C)$; and that the inscribed circle is the pedal of $(1, 1, 1)$, and so on.

The circle before us may also be looked upon as the auxiliary circle of the conic which touches α, β, γ, and has the points (ξ,η,ζ); $\left(\dfrac{1}{\xi},\dfrac{1}{\eta},\dfrac{1}{\zeta}\right)$ for foci.

The equation of this conic may be shewn to be

$$\sqrt{\xi(\eta^2 + 2\eta\zeta\cos A + \zeta^2)\alpha} + \sqrt{\eta(\zeta^2 + 2\zeta\xi\cos B + \xi^2)\beta}$$
$$+ \sqrt{\zeta(\xi^2 + 2\xi\eta\cos C + \eta^2)\gamma} = 0.$$

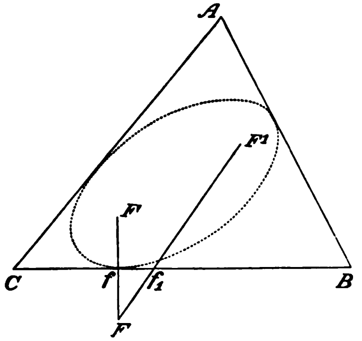

Fig. 9.

Let F, F' denote the points (ξ, η, ζ), $\left(\frac{1}{\xi}, \frac{1}{\eta}, \frac{1}{\zeta}\right)$, (see fig. 9); f the projection of F upon the side BC. Produce Ff to F_1 such that $FF_1 = 2Ff$, and join F_1 to the other focus F'; then the inscribed conic in question will touch BC at its point of intersection with F_1F'.

Now the co-ordinates of F, f are ξ, η, ζ, and

$$0,\ \eta + \xi\cos C,\ \zeta + \xi\cos B.$$

$\therefore$ Those of F_1 are given by the equations

$$\xi_1 + \xi = 0,$$

$$\eta_1 + \eta = 2(\eta + \xi\cos C),$$

$$\zeta_1 + \zeta = 2(\zeta + \xi\cos B);$$

which are equivalent to

$$\xi_1 = -\xi,$$

$$\eta_1 = \eta + 2\xi \cos C,$$

$$\zeta_1 = \zeta + 2\xi \cos B.$$

Hence the co-ordinates of the point where F_1F' meets BC will be found to be 0, $\eta(\zeta^2 + 2\zeta\xi \cos B + \xi^2)\beta = \zeta(\xi^2 + 2\xi\eta \cos C + \eta^2)\gamma$; and, therefore, the required equation is

$$\Sigma\sqrt{\xi(\eta^2 + 2\eta\zeta \cos A + \zeta^2) \,.\, \alpha} = 0. \qquad (3)$$

Either (1) or (2) represents, then, the auxiliary circle of the conic given by (3).

If $\dfrac{\xi}{1} = \dfrac{\eta}{\pm 1} = \dfrac{\zeta}{\pm 1}$; we have the equations, from (3), of the four circles which touch the sides.

If $\dfrac{\xi}{\cos A} = \dfrac{\eta}{\cos B} = \dfrac{\zeta}{\cos C}$, it follows that the circle which bisects the sides is the auxiliary circle of the conic

$$a\sqrt{\cos A\alpha} + b\sqrt{\cos B\beta} + c\sqrt{\cos C\gamma} = 0.$$

The equation to the radical axis of the circumscribed circle and of the pedal of (ξ, η, ζ) is seen from (2) to be

$$\Sigma\, \xi(\eta + \zeta \cos A)\,(\eta \cos A + \zeta) \sin B \sin C\alpha = 0.$$

If this passes through the fixed point (ξ_1, η_1, ζ_1) the locus of (ξ, η, ζ) will be the cubic

$$\Sigma \,.\, \frac{\xi_1}{a}\,(\eta + \zeta \cos A)\,(\eta \cos A + \zeta)\xi = 0.$$

If it touches the circumscribed circle we must have

$$\Sigma\sqrt{\xi\,(\eta + \zeta \cos A)\,(\eta \cos A + \zeta)} = 0\,;$$

or, the locus of a point (ξ, η, ζ) such that its pedal circle touches the circumscribing circle of the triangle of reference, is a curve of the sixth order touching the sides at the vertices, and also at the points where they are met by the radii AO, BO, CO, produced if necessary.

If the common chord in question passes through one of the foci, the locus of the other will be

$$\Sigma bc(\eta + \zeta\cos A)(\eta\cos A + \zeta) = 0,$$

or

$$bc\{(\eta^2 + \zeta^2)\cos A + \eta\zeta(1 + \cos^2 A)\}$$
$$+ ca\{(\zeta^2 + \xi^2)\cos B + \zeta\xi(1 + \cos^2 B)\}$$
$$+ ab\{(\xi^2 + \eta^2)\cos C + \xi\eta(1 + \cos^2 C) = 0,$$

which is equivalent to

$$a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 + bc(1 + \cos^2 A)\eta\zeta + ca(1 + \cos^2 B)\zeta\xi + ab(1 + \cos^2 C)\xi\eta = 0,$$

i.e. to

$$(a\xi + b\eta + c\zeta)^2 = bc\sin A(\sin A\,\eta\zeta + \sin B\,\zeta\xi + \sin C\,\xi\eta).$$

The locus is, therefore, an imaginary circle concentric with the circumscribed circle. I say *imaginary*, for its radius may be shewn to be $\sqrt{-3}\,.\,R$; R being the radius of the circle (ABC).

Again, if $\Sigma\sin A\,\eta\zeta = 0$; i.e. (ξ, η, ζ) lie on the circumscribed circle, equation (2) breaks up into the two factors

$$\Sigma\sin A\alpha, \text{ and } \Sigma\xi(\eta + \zeta\cos A)(\eta\cos A + \zeta)\frac{\alpha}{a}.$$

The latter expression, then, when equated to zero, represents in this case the line passing through the feet of the perpendiculars let fall from (ξ, η, ζ) upon the sides α, β, γ.

(For the equation of the envelope of this line subject to the above condition, viz. $\Sigma\sin A\,\eta\zeta = 0$, the reader may consult a paper by Mr. Greer, published in the "Quarterly Journal of Pure and Applied Mathematics," t. vii. p. 70.)

XVIII.

The pedal of (ξ, η, ζ), the circumscribed circle $\Sigma a\beta\gamma$, and the director of the inscribed conic

$$\Sigma \sqrt{\frac{\xi(\eta + \zeta\cos A)(\eta\cos A + \zeta)}{\cos A} \,.\, \alpha}, \text{ are co-axal.}$$

For the equation to the first circle is

$$\Sigma(1+x\cos A+\cos^2 A)\alpha^2 - \Sigma(2\cos B\cos C+x+y\cos C+z\cos B)\beta\gamma = 0;$$

where $\dfrac{\eta}{\zeta}+\dfrac{\zeta}{\eta}=x$; $\dfrac{\zeta}{\xi}+\dfrac{\xi}{\zeta}=y$; $\dfrac{\xi}{\eta}+\dfrac{\eta}{\xi}=z$.

Now let

$$\begin{aligned} 1+x\cos A+\cos^2 A &= \theta\,.\,L\cos A,\\ 1+y\cos B+\cos^2 B &= \theta\,.\,M\cos B,\\ 1+z\cos C+\cos^2 C &= \theta\,.\,N\cos C. \end{aligned}$$

Then

$$x = \theta L - \cos A - \frac{1}{\cos A},$$

$$y = \theta M - \cos B - \frac{1}{\cos B}, \quad z = \theta N - \cos C - \frac{1}{\cos C};$$

and the equation to the pedal becomes

$$\Sigma\theta L\cos A\alpha^2 - \Sigma\left\{\theta(L+M\cos C+N\cos B) - \left(\cos A+\sec A + \frac{\cos B}{\cos C}+\frac{\cos C}{\cos B}\right)\right\}\beta\gamma = 0,$$

or

$$\theta\,.\,\Sigma L\cos A\alpha^2 - \theta\Sigma(L+M\cos C+N\cos B)\beta\gamma + \tan A\tan B\tan C\,\Sigma\sin A\beta\gamma = 0;$$

since

$$\begin{aligned} &\cos A+\sec A+\frac{\cos B}{\cos C}+\frac{\cos C}{\cos B}\\ &= \cos A+\frac{1}{\cos A}+\frac{\sin^2 A-2\cos A\cos B\cos C}{\cos B\cos C}\\ &= \frac{\sin^2 A}{\cos A}+\frac{\sin^2 A}{\cos B\cos C}\\ &= \tan A\tan B\tan C\sin A. \end{aligned}$$

Hence, as $\Sigma L\cos A\alpha^2 - \Sigma(L + M\cos C + N\cos B)\beta\gamma = 0$ represents the director of the conic $\Sigma\sqrt{L\alpha}$, we have the theorem in question.

As a particular case of this, I may mention the following:—

The circle which bisects the sides, the director of the maximum inscribed ellipse, and the circumscribed circle are co-axal.

XIX.

The pedal circles of (ξ, η, ζ), (ξ', η', ζ'), and the director of the inscribed conic

$$\Sigma\,.\sqrt{\begin{vmatrix}\eta & \zeta \\ \eta' & \zeta'\end{vmatrix} \quad \begin{vmatrix}\eta & \zeta' \\ \zeta & \eta'\end{vmatrix}\xi\xi'\,.\,\alpha} = 0,$$

are co-axal.

For if $X = 0$, $X' = 0$ be the equations of the pedals in question,
$X = \Sigma(1 + x\cos A + \cos^2 A)\alpha^2$

$$- \Sigma(2\cos B\cos C + x + y\cos C + 2\cos B)\beta\gamma$$

$X' = \Sigma(1 + x'\cos A + \cos^2 A)\alpha^2 - \Sigma(2\cos B\cos C + x' + y'\cos C + z'\cos B)\beta\gamma,$

where $$\frac{\eta}{\zeta} + \frac{\zeta}{\eta} = x, \quad \frac{\eta'}{\zeta'} + \frac{\zeta'}{\eta'} = x', \text{ \&c.};$$

$$\therefore\ X - X' = \Sigma(x - x')\cos A\alpha^2$$

$$- \Sigma\{(x - x') + (y - y')\cos C + (z - z')\cos B\}\beta\gamma.$$

Now the right-hand side of this equation, when equated to zero, represents the locus of the point of intersection of rectangular tangents to the conic

$$\sqrt{(x - x')\alpha} + \sqrt{(y - y')\beta} + \sqrt{(z - z')\gamma} = 0.$$

Hence, since
$$x-x'=\frac{\eta}{\zeta}+\frac{\zeta}{\eta}-\frac{\eta'}{\zeta'}-\frac{\zeta'}{\eta'}$$
$$=\frac{\eta}{\zeta}-\frac{\eta'}{\zeta'}-\left(\frac{\zeta'}{\eta'}-\frac{\zeta}{\eta}\right)$$
$$=\frac{\eta\zeta'-\eta'\zeta}{\zeta\zeta'}-\frac{\eta\zeta'-\eta'\zeta}{\eta\eta'},$$
or
$$x-x'=\frac{(\zeta\eta'-\eta\zeta')(\eta\eta'-\zeta\zeta')}{\eta\eta'\zeta\zeta'}$$
$$y-y'=\text{\&c.},$$
we have the theorem enunciated above.

It is easily seen that the conic
$$\Sigma\sqrt{\begin{vmatrix}\eta & \zeta\\ \eta' & \zeta'\end{vmatrix}\begin{vmatrix}\eta & \zeta'\\ \zeta & \eta'\end{vmatrix}\xi\xi'\,.\,\alpha}$$
touches each of the four lines whose equations are
$$\Sigma\,.\begin{vmatrix}\eta & \zeta\\ \eta' & \zeta'\end{vmatrix}\alpha=0;\quad \Sigma\,.\begin{vmatrix}\eta & \zeta'\\ \zeta & \eta'\end{vmatrix}\xi\alpha=0$$
$$\Sigma\,.\begin{vmatrix}\eta & \zeta'\\ \zeta & \eta'\end{vmatrix}\xi'\alpha=0;\quad \Sigma\,.\begin{vmatrix}\eta & \zeta\\ \eta' & \zeta'\end{vmatrix}\xi\xi'\alpha=0:$$
i.e. that it touches the lines joining each of the points
$$(\xi,\eta,\zeta),\ \left(\frac{1}{\xi},\frac{1}{\eta},\frac{1}{\zeta}\right)\text{ to }(\xi',\eta',\zeta'),\ \left(\frac{1}{\xi'},\frac{1}{\eta'},\frac{1}{\zeta'}\right).$$

Hence we have the following proposition.

If p, p' denote the pair of inverse points (ξ,η,ζ), $\left(\frac{1}{\xi},\frac{1}{\eta},\frac{1}{\zeta}\right)$; and q, q' the pair (ξ',η',ζ'), $\left(\frac{1}{\xi'},\frac{1}{\eta'},\frac{1}{\zeta'}\right)$; then the seven lines α, β, γ, pq, pq', $p'q$, $p'q'$ are tangents to the same conic.

Consequences.

1. If p, for instance, be its own inverse, i.e. if it be the centre of any one of the four circles which touch the sides, then the tangents pq, $p'q$ (see fig. 10) will coincide,

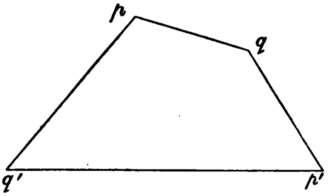

Fig. 10.

or, what is the same thing, two coincident tangents intersect at q; i.e. the curve will pass through q, and similarly through q'.

Hence, conversely, the four conics which touch the sides α, β, γ, and pass through any pair of inverse points, will have for tangents at each of these points the four lines joining it with the centres of the inscribed and escribed circles.

2. If q lie on the line pp', the seven-tangent conic in question will touch this line at q.

For example; Let G'—(a, b, c)—denote the inverse of the centroid $G - \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) -$ and p, p' coincide with the points O, P $(\cos A, \cos B, \cos C)$, $(\sec A, \sec B, \sec C)$; then the lines α, β, γ, OG, OG', PG' are tangents to a conic which passes through G.

The equation to this conic is

$$\Sigma \,.\, \cos A \sin (B - C) \sqrt{\sin A\alpha} = 0.$$

We see, then, that the theorem enunciated at the beginning of the section may be stated as follows:

The pedals of p or p', and of q or q'; the director of the conic which touches the seven lines $\alpha, \beta, \gamma, pq, pq', p'q, p'q'$, form three co-axal circles.

From this we may deduce the following propositions.

1°. The directors of the four conics which have for tangents the sides α, β, γ, and pass through the points O, P—$(\cos A, \cos B, \cos C)$, $(\sec A, \sec B, \sec C)$—touch respectively the nine-point circle at its points of contact with the inscribed and escribed circles.

This theorem is obtained at once by putting

$$\frac{\xi}{\cos A} = \frac{\eta}{\cos B} = \frac{\zeta}{\cos C}; \quad \frac{\xi'}{1} = \frac{\eta'}{\pm 1} = \frac{\zeta'}{\pm 1}.$$

2°. The following circles are co-axal.

The nine-point circle.—The pedal of the centroid G (i.e. the circle which trisects the distance between the middle point of each of the sides and the foot of the corresponding perpendicular). —The director of the conic which touches α, β, γ and the line PO at G.

3°. The nine-point circle.—The circle which bisects the segments between the middle points of the sides and the feet of the corresponding perpendiculars.—The director of the conic which touches the sides, and the segment PO at its middle point.

4°. If p be any point on the line PO, then the nine-point circle is the locus of the two points common to the pedal circle of p, and to the director of the inscribed conic which touches the line PO at the point in question.

This will be also true, if we substitute any pair of inverse points q, q' for P and O, and the pedal of q, or q' for the nine-point circle.

5°. Again, if p, q both lie on the circumscribed circle, their inverses will be on the line at infinity; so that the seven-tangent conic becomes a parabola, and the pedal circles become right lines as we may see from equation (2) of Sect. XVII.

Hence, if p, q be any two points on the circle (ABC), the point of intersection of their pedal lines will lie on the directrix of the parabola which touches $\alpha, \beta, \gamma, pq$; i.e. on the line which joins the polar centres of the triangles formed by these four tangents.

This may be otherwise stated as follows:—

Let Q denote any fixed parabola touching α, β, γ, and let pq be any other tangent to it, cutting the circle (ABC) in p and q; then the locus of the point of intersection of the pedal lines of p and q is the directrix of the parabola Q.

If p, q be the extremities of any chord of the circle (ABC) parallel to one of the sides of the triangle—to α—say, the locus of the point common to their pedal lines is the perpendicular AD drawn from A upon α.

(If p, q be the extremities of any *diameter* of the circle (ABC), it is known that their pedal lines intersect at right angles on the nine-point circle.

This elegant theorem was first given by Mr. Tucker in the "Educational Times," vol. iii. p. 58.)

6^0. Hence, also, if S denote the circle circumscribing the triangle —T—formed by any three out of four given right lines, it follows that the pedals (with respect to the sides of T) of the two points, where S is met by the fourth line, will intersect on the right line joining the polar centres of the four triangles of the given quadrilateral.

7^0. If I denote the centre of any one of the four circles which touch the sides, then the common pedal of any two inverse points p, p', the circle (I), and the director of the conic which touches the five lines α, β, γ, Ip, Ip' are co-axal.

As the result contained in theorem 1^0. is important, I will give a direct proof of it.

If we take the conic $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$, the equation of its director is

$$\begin{aligned} l\cos A\alpha^2 + m\cos B\beta^2 + n\cos C\gamma^2 - (l + m\cos C + n\cos B)\beta\gamma \\ - (m + n\cos A + l\cos C)\gamma\alpha \\ - (n + l\cos B + m\cos A)\alpha\beta = 0\,; \end{aligned}$$

which may be put under the form

$$\begin{aligned} (l\sin B\sin C + m\sin C\sin A + n\sin A\sin B)(\sin A\beta\gamma + \sin B\gamma\alpha + \sin C\alpha\beta) \\ = \Sigma\sin A\alpha \,.\, \Sigma l\sin B\sin C\cos A\alpha. \end{aligned}$$

Also the equation of the nine-point circle is

$$2(\sin A\beta\gamma + \sin B\gamma\alpha + \sin C\alpha\beta) = \Sigma\sin A\alpha \,.\, \Sigma\cos A\alpha\,;$$

hence that of their common chord is

$$\Sigma l \sin B \sin C \,.\, \Sigma \cos A\alpha = 2\,\Sigma \,.\, l \sin B \sin C \cos A\alpha,$$

or $$(-l \sin B \sin C + m \sin C \sin A + n \sin A \sin B) \cos A\alpha$$
$$+ (l \sin B \sin C - m \sin C \sin A + n \sin A \sin B) \cos B\beta$$
$$+ (l \sin C \sin A + m \sin C \sin A - n \sin A \sin B) \cos C\gamma = 0.$$

Now it is easily seen that for one of the four conics which pass through P, O and touch the lines α, β, γ, the ratios $l : m : n$ must

$$(\cos B - \cos C)^2 \cos A : (\cos C - \cos A)^2 \cos B : (\cos A - \cos B)^2 \cos C.$$

Hence the equation of the radical axis of the director of this conic and the nine-point circle is

$$\Sigma\{-\sin B \sin C \cos A (\cos B - \cos C)^2 + \sin C \sin A \cos B (\cos C - \cos A)^2$$
$$+ \sin A \sin B \cos C (\cos A - \cos B)^2\} \cos A\alpha = 0,$$

which will be found to reduce to

$$\frac{a\alpha}{b-c} + \frac{b\beta}{c-a} + \frac{c\gamma}{a-b} = 0;$$

since the co-efficient within the bracket is equivalent to

$$\frac{(a+b+c)^2}{4R^2bc}(a-b)(a-c)\cos B \cos C;$$

where $R =$ radius of circumscribing circle.

Hence, as the equation just found is known to represent the common tangent of the inscribed and nine-point circles, the theorem is proved.

It may also be shewn in the same way that the directors of the three conics

$$\sqrt{(\cos B - \cos C)^2 \cos A}\,\alpha + \sqrt{(\cos C + \cos A)^2 \cos B}\,.\,\beta$$
$$+ \sqrt{(\cos A + \cos B)^2 \cos C}\,.\,\gamma = 0,$$

$$\sqrt{(\cos B + \cos C)^2 \cos A}\,\alpha + \sqrt{(\cos C - \cos A)^2 \cos B}\,.\,\beta$$
$$+ \sqrt{(\cos A + \cos B)^2 \cos C}\,.\,\gamma = 0,$$

$$\sqrt{(\cos B + \cos C)^2 \cos A}\,\alpha + \sqrt{(\cos C + \cos A)^2 \cos B}\,.\,\beta$$
$$+ \sqrt{(\cos A - \cos B)^2 \cos C}\,.\,\gamma = 0,$$

touch the nine-point circle at its points of contact with the escribed circles, respectively.

(The equations of the four conics which touch α, β, γ, and pass through P and O, may be easily obtained by using tangential co-ordinates.

They are

$$(\lambda \cos A + \mu \cos B + \nu \cos C)(\lambda \sec A + \mu \sec B + \nu \sec C) = (\lambda \pm \mu \pm \nu)^2,$$

or
$$(\cos B - \cos C)^2 \cos A \, \mu\nu + (\cos C - \cos A)^2 \cos B \, \nu\lambda + (\cos A - \cos B)^2 \cos C \lambda\mu = 0,$$

$$(\cos B - \cos C)^2 \cos A \, \mu\nu + (\cos C + \cos A)^2 \cos B \, \nu\lambda + (\cos A + \cos B)^2 \cos C \lambda\mu = 0;$$

&c.

which are equivalent to the trilinear ones given above.)

XX.

If (I) denote the inscribed circle, (E_a) the circle which touches BC, say, and the two other sides produced, (ξ, η, η) any point on the line of centres of the two circles, then (I), (E_a), and the pedal of (ξ, η, η) are co-axal.

For the equation of the pedal of (ξ, η, ζ) is

$$\begin{aligned} X = {} & \Sigma(1 + \cos^2 A)\alpha^2 - \Sigma\, 2 \cos B \cos C \beta\gamma \\ & + x \cos A \alpha^2 + y \cos B \beta^2 + z \cos C \gamma^2 \\ & - (x + y \cos C + z \cos B)\beta\gamma \\ & - (y + z \cos A + x \cos C)\gamma\alpha \\ & - (z + x \cos B + y \cos A)\alpha\beta, \end{aligned}$$

where $x = \dfrac{\eta}{\zeta} + \dfrac{\zeta}{\eta}$; $y = \dfrac{\zeta}{\xi} + \dfrac{\xi}{\zeta}$; $z = \dfrac{\xi}{\eta} + \dfrac{\eta}{\xi}$.

If, then, $\zeta = \eta$ we have $x = 2$, $y = z$; and

$$\begin{aligned} X = \Sigma(1 + \cos^2 A)\alpha^2 - \Sigma\, 2\cos B\cos C\beta\gamma \\ + 2\cos A\alpha^2 + y(\cos B\beta^2 + \cos C\gamma^2) \\ - (2 + y\cos C + y\cos B)\beta\gamma \\ - (y + y\cos A + 2\cos C)\gamma\alpha \\ - (y + 2\cos B + y\cos A)\alpha\beta. \end{aligned}$$

Also,

$$\begin{aligned} (I) = \Sigma(1 + \cos^2 A)\alpha^2 - \Sigma\, 2\cos B\cos C\beta\gamma \\ + 2(\cos A\alpha^2 + \cos B\beta^2 + \cos C\gamma^2) \\ - 2(1 + \cos C + \cos B)\beta\gamma \\ - 2(1 + \cos A + \cos C)\gamma\alpha \\ - 2(1 + \cos B + \cos A)\alpha\beta. \end{aligned}$$

$$\begin{aligned} (E_a) = \Sigma(1 + \cos^2 A)\alpha^2 - \Sigma\, 2\cos B\cos C\beta\gamma \\ + 2(\cos A\alpha^2 - \cos B\beta^2 - \cos C\gamma^2) \\ - 2(1 - \cos C - \cos B)\beta\gamma \\ - 2(-1 - \cos A + \cos C)\gamma\alpha \\ - 2(-1 + \cos B - \cos A)\alpha\beta; \end{aligned}$$

$$\begin{aligned} \therefore X - (I) = (y - 2)\{\cos B\beta^2 + \cos C\gamma^2 - (\cos B + \cos C)\beta\gamma \\ - (1 + \cos A)\gamma\alpha - (1 + \cos A)\alpha\beta\}, \end{aligned}$$

and

$$\begin{aligned} (I) - (E_a) = 4\{\cos B\beta^2 + \cos C\gamma^2 - (\cos B + \cos C)\beta\gamma - (1 + \cos A) \\ (\gamma\alpha + \alpha\beta)\}, \end{aligned}$$

$$\therefore 4\{X - (I)\} = (y - 2)\{(I) - E_a)\},$$

or $$4X = (y + 2)(I) - (y - 2)(E_a),$$

which proves the theorem.

Hence, if I_1 be the point where the internal bisector AI cuts the side BC, the circles (I), (E_a), and the circle described upon the segment AI_1 as diameter are co-axal.

XXI.

To find the condition that the pedal of (ξ, η, ζ) shall touch the inscribed circle (I).

The equation of the pedal in question being (see Equation (2), Sect. XVII.)

$$X = \Sigma \sin A a \, . \, \Sigma \xi(\eta + \zeta \cos A)(\eta \cos A + \zeta) \sin B \sin C a$$
$$- \Sigma \sin A \, \xi \, . \, \Sigma \sin A \, \eta\zeta \, . \, \Sigma \sin A \beta\gamma,$$

we have

$$\left\{ \frac{X}{\Sigma \sin A\xi \, . \, \Sigma \sin A\eta\zeta} - \frac{I}{(\sin A + \sin B + \sin C)^2} \right\}$$

$$= \Sigma \sin A a \, . \left\{ \frac{\Sigma \xi(\eta + \zeta \cos A)(\eta \cos A + \zeta) \sin B \sin C a}{\Sigma \sin A\xi \, . \, \Sigma \sin A\eta\zeta} - \frac{\Sigma (1 + \cos A)^2 \sin B \sin C a}{(\sin A + \sin B + \sin C)^2} \right\};$$

hence the radical axis of X and I is given by the equation

$$(\sin A + \sin B + \sin C)^2 \, \Sigma \xi (\eta + \zeta \cos A)(\eta \cos A + \zeta) \sin B \sin C a$$
$$- \Sigma \sin A\xi \, . \, \Sigma \sin A \, \eta\zeta \, . \, \Sigma (1 + \cos A)^2 \sin B \sin C a = 0:$$

and the condition that this line shall touch I is

$$\frac{(1 + \cos A) \sin A}{(\sin A + \sin B + \sin C)^2 \, \xi(\eta + \zeta \cos A)(\eta \cos A + \zeta) - (1 + \cos A)^2 \, \Sigma \sin A\xi \, . \, \Sigma \sin A\eta\zeta}$$
$$+ \frac{(1 + \cos B) \sin B}{(\sin A + \sin B + \sin C)^2 \, \eta(\zeta + \xi \cos B)(\zeta \cos B + \xi) - (1 + \cos B)^2 \, \Sigma \sin A\xi \, . \, \Sigma \sin A\eta\zeta}$$
$$+ \frac{(1 + \cos C) \sin C}{(\sin A + \sin B + \sin C)^2 \, \zeta(\xi + \eta \cos C)(\xi \cos C + \eta) - (1 + \cos C)^2 \, \Sigma \sin A\xi \, . \, \Sigma \sin A\eta\zeta}$$
$$= 0;$$

or the locus of the foci of an inscribed conic such that its auxiliary circle touches the inscribed circle is the curve of the sixth order given by the equation

$$\frac{\tan\frac{A}{2}}{\Sigma\sin A\xi\,.\,\Sigma\sin A\eta\zeta-\frac{4}{\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}\,.\,\xi(\eta+\zeta\cos A)(\eta\cos A+\zeta)}$$

$$+\frac{\tan\frac{B}{2}}{\Sigma\sin A\xi\,.\,\Sigma\sin A\eta\zeta-\frac{4}{\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)^2}\,.\,\eta(\zeta+\xi\cos B)(\zeta\cos B+\xi)}$$

$$+\frac{\tan\frac{C}{2}}{\Sigma\sin A\xi\,.\,\Sigma\sin A\eta\zeta-\frac{4}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^2}\,.\,\zeta(\xi+\eta\cos C)(\xi\cos C+\eta)}$$

$$=0:$$

which, if

$$U=\Sigma\sin A\xi\,.\,\Sigma\sin A\eta\zeta-\frac{4\,\xi(\eta+\zeta\cos A)(\eta\cos A+\zeta)}{\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}$$

$$V=\Sigma\sin A\xi\,.\,\Sigma\sin A\eta\zeta-\frac{4\,\eta(\zeta+\xi\cos B)(\zeta\cos B+\xi)}{\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)^2}$$

$$W=\Sigma\sin A\xi\,.\,\Sigma\sin A\eta\zeta-\frac{4\,\zeta(\xi+\eta\cos C)(\xi\cos C+\eta)}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^2};$$

may be put under the form $\Sigma\tan\frac{A}{2}\,VW=0$.

XXII.

The circumscribing circle (ABC), the nine-point circle $(A_1B_1C_1)$, and the polar circle are co-axal.

This may be proved by simply writing down the equations of these respective circles, viz.,

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$$

$$(\cos A\alpha + \cos B\beta + \cos C\gamma)(a\alpha + b\beta + c\gamma) - 2(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0$$

$$(\cos A\alpha + \cos B\beta + \cos C\gamma)(a\alpha + b\beta + c\gamma) - (a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0.$$

If t_1, t_2, t_3 denote the points where the tangents to the circle (ABC) at the vertices A, B, C cut the opposite sides BC, CA, AB respectively, the radical axis $\Sigma \cos A\alpha = 0$, passes through the middle points of the segments At_1, Bt_2, Ct_3.

For the co-ordinates of the middle point of At_1, for instance, are

$$\sin B \sin C, \ \frac{\sin^2 B \sin C}{\sin(B - C)}, \ \frac{\sin^2 C \sin B}{\sin(C - B)};$$

and the condition that it shall lie on the line $\Sigma \cos A\alpha$, is

$$\cos A + \cos(B + C) = 0, \text{ an identity.}$$

The radical axis in question is also, evidently, the axis of perspective of the triangles ABC, DEF.

Consequences.

1°. If P—as before—denote the polar centre; and if O, the centre of the circumscribed circle, be joined to each of the points of bisection of the sides, A_1, B_1, C_1, and OA_1, OB_1, OC_1, be produced to O_1, O_2, O_3, so that $OO_1 = 2OA_1$; $OO_2 = 2OB_1$; $OO_3 = 2OC_1$, it follows that the nine-point circle passes through the two points common to the polar and circumscribing circles of each of the triangles, PBC, PCA, PAB, ABC, OO_2O_3, OO_3O_1, OO_1O_3, $O_1O_2O_3$.

2°. If the point of intersection of any two lines xy, $x'y'$ be denoted by $(xy, x'y')$, the nine-point circle $(A_1B_1C_1)$ passes through the intersections of the nine-point and circumscribing circles of the triangle whose vertices are (B_1C_1, EF), (C_1A_1, FD), and (A_1B_1, DE). (See explanation of the Notation).

This theorem will be proved if we can shew that the triangle in question is self-conjugate with respect to the circle $(A_1B_1C_1)$, which may be done as follows:

The equations of the sides of the common self-conjugate triangle of two conics can be found at once by forming the Jacobian of these conics and the co-variant F. (See Dr. Salmon's "Conic Sections," 4th edit., p. 344, Ex. 2.)

Supposing, then, that we take the conics

$$S = \Sigma l^2\alpha^2 - 2\,\Sigma mn\beta\gamma,$$

$$S' = \Sigma l'^2\alpha^2 - 2\,\Sigma m'n'\beta\gamma,$$

$$F = \Sigma ll'\alpha^2 - \Sigma(mn' + m'n)\beta\gamma,$$

their Jacobian is easily found to be

$$J = \begin{vmatrix} l(-l\alpha + m\beta + n\gamma), & m(l\alpha - m\beta + n\gamma), & n(l\alpha + m\beta - n\gamma) \\ l'(-l'\alpha + m'\beta + n'\gamma), & m'(l'\alpha - m'\beta + n'\gamma), & n'(l'\alpha + m'\beta - n'\gamma) \\ \dfrac{dF}{d\alpha}, & \dfrac{dF}{d\beta}, & \dfrac{dF}{d\gamma} \end{vmatrix};$$

now this determinant vanishes when $-l\alpha + m\beta + n\gamma = 0$, and $-l'\alpha + m'\beta + n'\gamma = 0$, simultaneously, for then it reduces to

$$\begin{vmatrix} 0, & 2mn\gamma, & 2mn\beta \\ 0, & 2m'n'\gamma, & 2m'n'\beta \\ \dfrac{dF}{d\alpha}, & \dfrac{dF}{d\beta} & \dfrac{dF}{d\gamma} \end{vmatrix}, \text{ which is, obviously, } = 0.$$

Hence, as we know that the Jacobian in question breaks up into three right lines, they must be those which join the three points given by the following pairs of equations:

$$\left.\begin{array}{r} -l\alpha+m\beta+n\gamma=0 \\ -l'\alpha+m'\beta+n'\gamma=0 \end{array}\right\};\quad \left.\begin{array}{r} l\alpha-m\beta+n\gamma=0 \\ l'\alpha-m'\beta+n'\gamma=0 \end{array}\right\};$$

$$\left.\begin{array}{r} l\alpha+m\beta-n\gamma=0 \\ l'\alpha+m'\beta-n'\gamma=0 \end{array}\right\}.$$

i.e. if x, y, z, x', y', z' be the points where S, S' touch the respective sides of the triangle of reference, the common self-conjugate triangle of S, S', and F is that whose vertices are the points $(yz, y'z')$, $(zx, z'x')$, $(xy, x'y')$.

If $\dfrac{l}{\sin A}=\dfrac{m}{\sin B}=\dfrac{n}{\sin C}$, $\quad\dfrac{l'}{\cos A}=\dfrac{m'}{\cos B}=\dfrac{n'}{\cos C}$,

we have the theorem which was required to be proved.

XXIII.

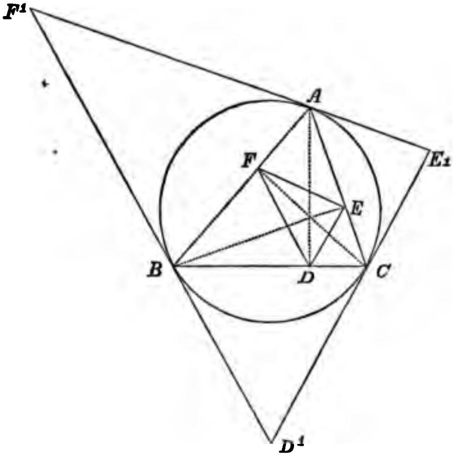

Fig. 11.

If D', E', F' denote the vertices of the triangle formed by the tangents at A, B, C to the circle (ABC)—see fig. (11)—the circles (ABC), (DEF), $(D'E'F')$ are co-axal.

For the co-ordinates of the points D', E', F' are readily found to be

$$\begin{array}{ccc} -\sin A, & \sin B, & \sin C, \\ \sin A, & -\sin B, & \sin C, \\ \sin A, & \sin B, & -\sin C; \end{array}$$

which lie on the circle whose equation is

$$(\sin A\alpha+\sin B\beta+\sin C\gamma)(\cos A\alpha+\cos B\beta+\cos C\gamma) \\ +4\cos A\cos B\cos C(\sin A\beta\gamma+\sin B\gamma\alpha+\sin C\alpha\beta)=0.$$

Consequences.

If $\alpha_1, \beta_1, \gamma_1$ denote the points where the sides BC, CA, AB are met by any one of the four circles which touch them—see fig. (12)—

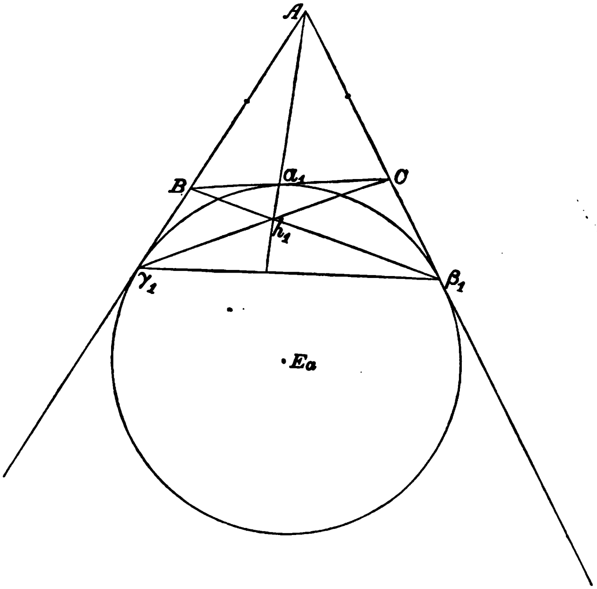

Fig. 12.

it follows that the circle ABC passes through the two points common to the circumscribing and nine-point circle of the triangle $\alpha_1\beta_1\gamma_1$.

These points may be shewn to be the foci of the two parabolas which can be drawn through A, B, C, and the centre of perspective of the triangles ABC, $\alpha_1\beta_1\gamma_1$.

For if $\alpha_1\beta_1\gamma_1$ be taken as the triangle of reference, and if a_1, b_1, c_1 be the lengths of the sides, the co-ordinates of A, B, C and of the centre of perspective in question are $-a_1, b_1, c_1$; $a_1, -b_1, c_1$; $a_1, b_1, -c_1$; a_1, b_1, c_1, and, therefore, the two parabolas which circumscribe them will be represented by

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

where
$$\frac{a_1^2}{l} + \frac{b_1^2}{m} + \frac{c_1^2}{n} = 0,$$

and
$$a_1^2 l + b_1^2 m + c_1^2 n = 0.$$

Again, if f, g, h be the co-ordinates of the focus of either of these curves, we have

$$\frac{l}{-a_1f + b_1g + c_1h} = \frac{m}{a_1f - b_1g + c_1h} = \frac{n}{a_1f + b_1g - c_1h},$$

—(see Mr. Ferrers' "Trilinear Co-ordinates," p. 116)—

or the foci are the points of intersection of the line

$$a_1^2(-a_1f+b_1g+c_1h) + b_1^2(a_1f-b_1g+c_1h) + c_1^2(a_1f+b_1g-c_1h) = 0,$$

with the conic

$$\frac{a_1^2}{-a_1f + b_1g + c_1h} + \frac{b_1^2}{a_1f - b_1g + c_1h} + \frac{c_1^2}{a_1f + b_1g - c_1h} = 0;$$

i.e. of the line

$$\cos A_1\alpha + \cos B_1\beta + \cos C_1\gamma = 0,$$

with the nine-point circle

$$\Sigma a_1 \cos A_1 \alpha^2 - \Sigma a_1 \beta\gamma = 0,$$

which proves the proposition.

Again, since the circle (ABC), represented by the equation

$$\Sigma \sin A_1 \alpha \,.\, \Sigma \cos A_1 \alpha + 4 \cos A_1 \cos B_1 \cos C_1 \,\Sigma \sin A_1 \beta\gamma = 0,$$

passes through the two points in question, we have the following theorem.

Let $\alpha_1, \beta_1, \gamma_1$ be the points where the sides of a given triangle ABC, are met by one of the four circles which touch its sides; also let h_1 be the point of intersection of the lines $A\alpha_1$, $B\beta_1$, $C\gamma_1$; then the foci of the parabolas which pass through A, B, C, h_1 will lie on the circle (ABC); being, in fact, the two points where this circle is intersected by the circle $(\alpha_1\beta_1\gamma_1)$. See Fig. 13, (p. 46.)

Hence, if D, E, F, as before, denote the feet of the perpendiculars of any triangle ABC; $\alpha_1, \beta_1, \gamma_1$ the projections of one of the vertices A, B, C upon the sides EF, FD, DE; h_1 the centre of perspective of the triangles $\alpha_1\beta_1\gamma_1$, DEF; then the foci of the two parabolas which pass through D, E, F and h_1 will lie on the nine-point circle of the triangle ABC.

XXIV.

It may be observed that the triangles DEF, $D'E'F'$ are similar and in perspective—see fig. 11.

For, if we take the side $E'F'$, for instance, its equation is

$$\frac{\beta}{\sin B} + \frac{\gamma}{\sin C} = 0,$$

or, what is the same thing,

$$\cos A(\sin A\alpha + \sin B\beta + \sin C\gamma) + \sin A(-\cos A\alpha + \cos B\beta + \cos C\gamma) = 0\,;$$

i.e. $E'F'$ and EF are parallel.

Hence, if d, e, f denote the feet of the perpendiculars of the

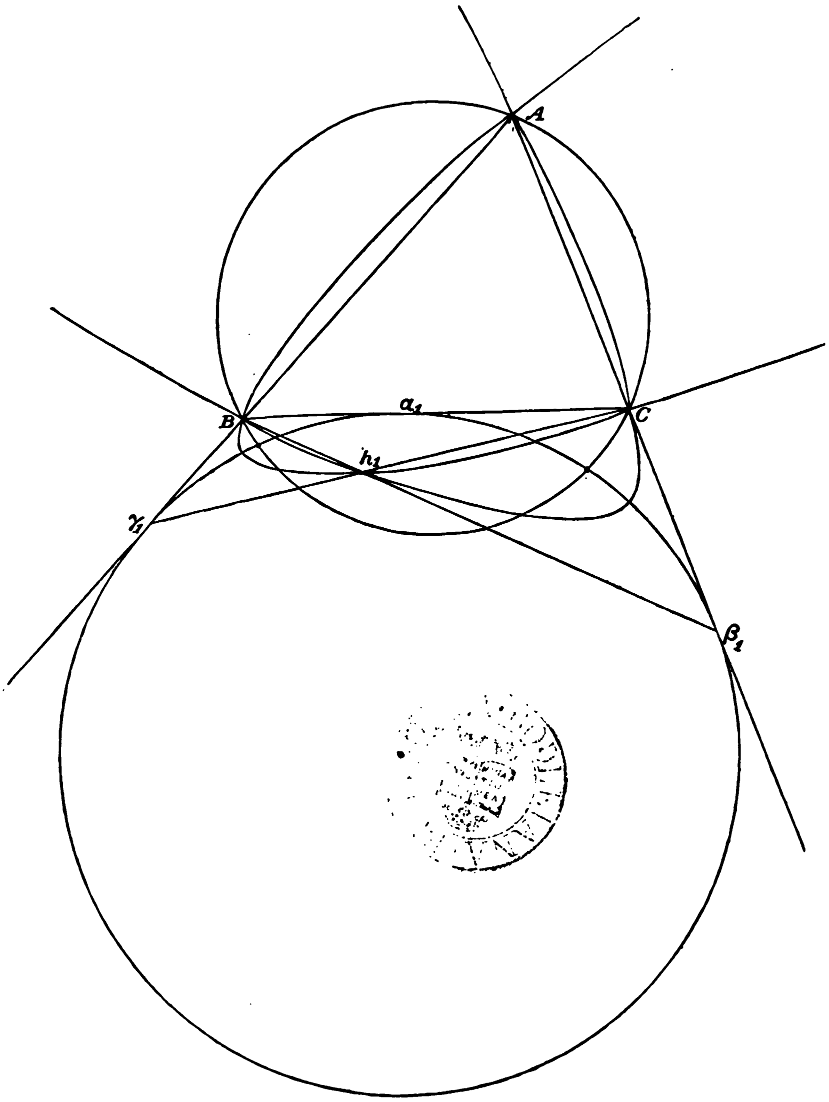

Fig. 13.

triangle $\alpha_1\beta_1\gamma_1$—see fig. 12—the line ef, fd, de will be parallel to the sides of the triangle ABC.

Again, since the lines DD', FE', FF' are represented by the equations

$$\cos A\sin(B-C)\alpha - \sin A(\cos B\beta - \cos C\gamma) = 0,$$

$$\cos B\sin(C-A)\beta - \sin B(\cos C\gamma - \cos A\alpha) = 0,$$

$$\cos C\sin(A-B)\gamma - \sin C(\cos A\alpha - \cos B\beta) = 0;$$

each of which is satisfied by the values

$$\frac{\cos A}{\sin^2 A}\cdot\alpha = \frac{\cos B}{\sin^2 B}\cdot\beta = \frac{\cos C}{\sin^2 C}\cdot\gamma,$$

it follows that the two triangles are in perspective.

If p_1 denote the centre of perspective in question, it may be shewn that the circle which has for a diameter the line joining p_1 to the centre of the circle (ABC) passes through the intersections of the circles (DEF), $(D'E'F')$.

Now the equation $\Sigma\sin A\alpha \,.\, \Sigma\cos A\alpha = k\,\Sigma\sin A\beta\gamma$ will represent a circle passing through $O(\cos A, \cos B, \cos C)$ provided that

$$k = 2(1 - 2\cos A\cos B\cos C);$$

but this is also the condition that the same circle shall pass through

$$p_1\left(\frac{\sin^2 A}{\cos A}, \frac{\sin^2 B}{\cos B}, \frac{\sin^2 C}{\cos C}\right);$$

for if these co-ordinates be substituted in the equation before us, we have

$$k = \frac{\Sigma\dfrac{\sin^3 A}{\cos A}\,.\,\Sigma\sin^2\mathrm{A}}{\Sigma\left(\dfrac{\sin A\sin^2 B\sin^2 C}{\cos B\cos C}\right)}$$

$$= \frac{\Sigma\sin^3 A\cos B\cos C\,.\,\Sigma\sin^2 A}{\sin A\sin B\sin C\,.\,\Sigma\sin B\sin C\cos A};$$

or since

$$\Sigma \sin^3 A \cos B \cos C = \Sigma \sin A \cos B \cos C - \cos A \cos B \cos C \, \Sigma \sin A \cos A,$$

$$= \sin A \sin B \sin C (1 - 2 \cos A \cos B \cos C);$$

and $$\Sigma \sin^2 A = 2 \Sigma \sin B \sin C \cos A,$$

$$k = 2(1 - 2 \cos A \cos B \cos C).$$

Hence, as each of the points

$$(\cos A, \cos B, \cos C); \left(\frac{\sin^2 A}{\cos A}, \frac{\sin^2 B}{\cos B}, \frac{\sin^2 C}{\cos C}\right)$$

lie on the line of centres of the system of circles represented by

$$\Sigma \sin A\alpha \,.\, \Sigma \cos A\alpha = k \Sigma \sin A\beta\gamma,$$

the theorem is proved.

It may be also shewn that the director of the conic with respect to which the triangle ABC is self-conjugate, and $D'E'F'$ is the polar triangle of DEF, will pass through the intersections of the circles (DEF), $(D'E'F')$.

For the equation to the conic in question is easily found to be

$$\cot A\alpha^2 + \cot B\beta^2 + \cot C\gamma^2 = 0;$$

and that of its director to be

$$\Sigma a \cos A\alpha^2 + 2 \cos A \cos B \cos C \,.\, \Sigma a\beta\gamma = 0,$$

or $$\Sigma \sin A\alpha \,.\, \Sigma \cos A\alpha = (1 - 2 \cos A \cos B \cos C) \Sigma \sin A\beta\gamma,$$

whence the truth of the theorem is at once seen.

It may be observed that this director is orthogonal to the circle (ABC), and that its centre coincides with the centre of perspective of the triangles DEF, $D'E'F'$.

XXV.

The polar circles of the four triangles formed by the lines *BC*, *CA*, *AB*, and the axis of homology of the triangles *ABC*, *D′E′F′* —see Fig. (11)—all pass through the two points common to the circles (*ABC*), (*DEF*), (*D′E′F′*).

If the triangle *ABC* be taken as the triangle of reference, the equations to the four lines in question are

$$\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \text{and}\ \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0\,;$$

and we have to shew that the equation to the polar circle of the triangle formed by β, γ, $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}$, for instance, admits of being put under the form

$$\Sigma a \cos A\alpha^2 + k\,\Sigma a\beta\gamma = 0.$$

Now the polar of the point *A* with respect to this circle is given by the equation

$$2\,a \cos A\alpha + k\,(c\beta + b\gamma) = 0,$$

which will coincide with $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0$, if $k = \frac{2a^2}{bc}\cos A$.

Also the equation of the polar of $\left(\beta = 0,\ \frac{\alpha}{a} + \frac{\gamma}{c} = 0\right)$ is

$$a\{2a \cos A\alpha + k(c\beta + b\gamma)\} - c\{2c \cos C\gamma + k(b\alpha + a\beta)\} = 0,$$

or $$(2a^2 \cos A - kbc)\alpha + (abk - 2c^2 \cos C)\gamma = 0,$$

which coincides with $\gamma = 0$, if $k = \frac{2a^2}{bc}\cos A$.

Hence the polar circle of the triangle formed by the lines β, γ, and $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}$, is represented by the equation

$$\Sigma a \cos A\alpha^2 + \frac{2a^2}{bc}\cos A\ \Sigma a\beta\gamma = 0,$$

similarly, those of the other triangles will be given by

$$\Sigma a \cos A\alpha^2 + \frac{2b^2}{ca} \cos B \, \Sigma \alpha\beta\gamma = 0,$$

$$\Sigma a \cos A\alpha^2 + \frac{2c^2}{ab} \cos C \, \Sigma \alpha\beta\gamma = 0,$$

$$\Sigma a \cos A\alpha^2 = 0,$$

whence the truth of the theorem is manifest.

Again, since the polar circle of any triangle circumscribing a rectangular hyperbola passes through the centre of the curve, it follows that the two points common to the above system are the centres of the two rectangular hyperbolas which touch the four lines α, β, γ, $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}$.

(These points have already been shewn to be the foci of the two parabolas which pass through $(-a, b, c)$; $(a, -b, c)$; $(a, b, -c)$; (a, b, c).).

Consequences.

1. If t_1, t_2, t_3 be the points where the line $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}$ cuts the sides BC, CA, AB; the circle which has for a diameter any one of the segments At_1, Bt_2, Ct_3—or generally—the director of any conic touching α, β, γ, $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}$, cuts orthogonally each of the above system of polar circles.

2. The radical axis of the directors in question coincides with the line PO, (see explanation of the Notation), which passes through the polar centre of each of the four triangles of the above quadrilateral.

3. The radical axis of the above system of polar circles bisects the segments At_1, Bt_2, Ct_3.

4. The line PO is the directrix of the parabola which touches $\alpha, \beta, \gamma, \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}$.

RÉSUMÉ.

If P denote the point of intersection of the three perpendiculars, O the centre of the circumscribing circle; G the centre of gravity of the area of any triangle ABC; D, E, F the feet of the three perpendiculars; A_1, B_1, C_1 the middle points of the sides; D', E', F' the vertices of the triangle formed by the tangents at A, B, C to the circle (ABC); it will be seen on reference to the preceding sections that the following circles are co-axal: viz.,

1°. The circumscribing circle (ABC).

2°. The circle $(A_1B_1C_1)$ or (DEF).

3°. The polar circles of the four triangles of the quadrilateral $\alpha, \beta, \gamma, \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}$.

4°. The circle $(D'E'F')$.

5°. The circle whose diameter is the line joining O to p_1—the centre of perspective of the triangles DEF, $D'E'F'$.

6°. The circle described with centre p_1 to cut the circle (ABC) orthogonally.

7°. The director of the maximum ellipse that can be inscribed in the triangle ABC.

8°. The circle which passes through the six points of intersection of the three pairs of circles (AA_1), (PBC); (BB_1), (PCA); (CC_1), (PAB)—see Sect. IX.

9°. The circle described upon the line PG as diameter.

The equations of these respective circles are,

$$\Sigma a\beta\gamma = 0,$$

$$\Sigma \cos A\alpha \,.\, \Sigma a\alpha - 2\,\Sigma a\beta\gamma = 0,$$

$$\left.\begin{array}{r}\Sigma \cos A\alpha \,.\, \Sigma a\alpha - \Sigma a\beta\gamma = 0, \\ \Sigma \cos A\alpha \,.\, \Sigma a\alpha - \left(1 - \dfrac{a^2}{b^2}\right)\left(1 - \dfrac{a^2}{c^2}\right)\Sigma a\beta\gamma = 0, \\ \Sigma \cos A\alpha \,.\, \Sigma a\alpha - \left(1 - \dfrac{b^2}{c^2}\right)\left(1 - \dfrac{b^2}{c^2}\right)\Sigma a\beta\gamma = 0, \\ \Sigma \cos A\alpha \,.\, \Sigma a\alpha - \left(1 - \dfrac{c^2}{a^2}\right)\left(1 - \dfrac{c^2}{b^2}\right)\Sigma a\beta\gamma = 0, \end{array}\right\}$$

$$\Sigma \cos A\alpha \,.\, \Sigma a\alpha + 4\cos A\cos B\cos C\,\Sigma a\beta\gamma = 0,$$

$$\Sigma \cos A\alpha \,.\, \Sigma a\alpha - 2(1 - 2\cos A\cos B\cos C)\,\Sigma a\beta\gamma = 0,$$

$$\Sigma \cos A\alpha \,.\, \Sigma a\alpha - (1 - 2\cos A\cos B\cos C)\,\Sigma a\beta\gamma = 0,$$

$$\Sigma \cos A\alpha \,.\, \Sigma a\alpha - 3\,\Sigma a\beta\gamma = 0,$$

$$2\,\Sigma \cos A\alpha \,.\, \Sigma a\alpha - 5\,\Sigma a\beta\gamma = 0,$$

$$2\,\Sigma \cos A\alpha \,.\, \Sigma a\alpha - 3\,\Sigma a\beta\gamma = 0.$$

The values of k corresponding to the limiting points of the system

$$\Sigma a\cos A\alpha^2 + k\,\Sigma a\beta\gamma = 0,$$

are given by the quadratic

$$k^2 = 4(1 + k)\cos A\cos B\cos C; \qquad \text{(see Sect. VIII.)},$$

or

$$k = 2\cos A\cos B\cos C \pm 2\sqrt{\cos A\cos B\cos C\,(1 + \cos A\cos B\cos C)}.$$

Hence, these points will be real, coincident, or imaginary, according as $\cos A \cos B \cos C$ is positive, zero, or negative; i.e. according as the triangle is acute-, right-, or obtuse-angled.

XXVI.

The director of any conic which touches α, β, γ and passes through O, the centre of the circumscribing circle, has contacts of a similar species with this circle and the nine-point circle.

If the conic $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$, passes through O, we have

$$\sqrt{l\cos A} + \sqrt{m\cos B} + \sqrt{n\cos C} = 0. \tag{1}$$

Also the equation of its director is

$$\Sigma l\cos A\alpha^2 - \Sigma(l + m\cos C + n\cos B)\beta\gamma = 0\ ;$$

or, if

$$S_a = \cos A\alpha^2 - \beta\gamma - \cos C\gamma\alpha - \cos B\alpha\beta,$$

$$S_b = \cos B\beta^2 - \gamma\alpha - \cos A\alpha\beta - \cos C\beta\gamma,$$

$$S_c = \cos C\gamma^2 - \alpha\beta - \cos B\beta\gamma - \cos A\gamma\alpha,$$

it becomes

$$lS_a + mS_b + nS_c = 0. \tag{2}$$

We have, then, to find the envelope of (2), when the variables l, m, n are subject to the condition (1).

Differentiating (1) and (2) we have

$$\sqrt{\frac{\cos A}{l}}\,.\,dl + \sqrt{\frac{\cos B}{m}}\,.\,dm + \sqrt{\frac{\cos C}{n}}\,.\,dn = 0,$$

$$S_a\,.\,dl + S_b\,.\,dm + S_c\,.\,dn = 0,$$

$$\therefore\ \frac{\sqrt{\dfrac{\cos A}{l}}}{S_a} = \frac{\sqrt{\dfrac{\cos B}{m}}}{S_b} = \frac{\sqrt{\dfrac{\cos C}{n}}}{S_c},$$

and the equation to the required envelope is

$$\frac{\cos A}{S_a} + \frac{\cos B}{S_b} + \frac{\cos C}{S_c} = 0,$$

or

$$\cos A\,.\,S_bS_c + \cos B\,.\,S_cS_a + \cos C\,.\,S_aS_b = 0.$$

Now, if S_a, S_b, S_c be multiplied by $\sin A$, $\sin B$, $\sin C$ respectively, they can be put under the form

$$\sin A \,.\, S_a = \alpha \cos A \,.\, L - S,$$

$$\sin B \,.\, S_b = \beta \cos B \,.\, L - S,$$

$$\sin C \,.\, S_c = \gamma \cos C \,.\, L - S;$$

where
$$L = \sin A\alpha + \sin B\beta + \sin C\gamma,$$

$$S = \sin A\beta\gamma + \sin B\gamma\alpha + \sin C\alpha\beta;$$

hence the above equation becomes

$$\Sigma \sin A \cos A \left\{\cos B \cos C\beta\gamma \,.\, L^2 - (\beta \cos B + \gamma \cos C) \,.\, SL + S^2\right\} = 0,$$

or
$$\cos A \cos B \cos C \,.\, SL^2 - \Sigma \sin A \cos A \,(\beta \cos B + \gamma \cos C) \,.\, SL + \Sigma \sin A \cos A \,.\, S^2 = 0,$$

which breaks up into the two factors S, and

$$2S - (\cos A\alpha + \cos B\beta + \cos C\gamma) L;$$

hence the theorem in question.

Again, since the director of any conic touching α, β, γ cuts orthogonally the polar circle—see Sect. V.—it follows from the converse of a known theorem—("Every circle having contacts of similar species with two others intersects at right angles the co-axal circle whose centre is their external centre of perspective." —"Modern Geometry," vol. i. p. 290)—that the contacts are of similar species.

Hence the lines joining the points of contacts all pass through the polar centre P, which is the external centre of similitude of the nine-point and circumscribing circles.

Consequences.

1°. The nine-point circle is touched by the director of any conic which bisects the segment PO and touches the sides of either of the triangles $A_1B_1C_1$, DEF.—(See explanation of the Notation.)

Also since the triangles PBC, PCA, PAB, OO_1O_2, OO_2O_3, OO_3O_1, $O_1O_2O_3$ (see Sect. XXII.) have their sides bisected by one and the same circle, it follows that this must touch an indefinite number of directors.

2°. The directors of the four conics which pass through P, O and touch the sides α, β, γ, have contacts of similar species with the nine-point and circumscribing circles.

(It has been already shewn that these directors touch the nine-point circle at its points of contact with the inscribed and escribed circles.)

3°. If the inscribed circle passes through the centre of the circumscribing circle, the director of the former will touch both the latter and the nine-point circle.

This may be easily proved from other considerations, as follows: If $\delta =$ distance between the centres of the circumscribed and inscribed circles, then we know that $\delta^2 = R^2 - 2Rr$; hence since δ is by hypothesis $= r$, we have

$$R^2 - 2Rr = r^2$$

$$(R - r)^2 = 2r^2,$$

or

$$\delta = r = R - r\sqrt{2}.$$

Again, if $\delta^1 =$ distance between the inscribed and nine-point circles $\delta^1 = \frac{R}{2} - r$,

or since

$$R - r = r\sqrt{2}$$

$$\delta^1 = r\sqrt{2} - \frac{R}{2}.$$

4°. If a_1, b_1, c_1 be the points where the radii AO, BO, CO—produced, if necessary,—cut the opposite sides BC, CA, AB; the circles whose diameters are the segments Aa_1, Bb_1, Cc_1 will each touch the nine-point circle, the points of contact being the feet of the three perpendiculars.

This may be otherwise proved, as follows:—

The tangential equation of the circle (Aa_1) is easily found to be

$$\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C$$
$$= \{\lambda\cos(B - C) + \mu\cos B + \nu\cos C\}^2,$$

and, therefore, one of the centres of similitude of this circle and the nine-point circle

$$\lambda^2+\mu^2+\nu^2-2\mu\nu\cos A-2\nu\lambda\cos B-2\lambda\mu\cos C$$
$$=\{\lambda\cos(B-C)+\mu\cos(C-A)+\nu\cos(A-B)\}^2,$$

is given by the equation

$$\frac{\mu}{\cos B}+\frac{\nu}{\cos C}=0;$$

which represents the foot of the perpendicular from A upon the side BC—.

Hence the two circles in question will touch each other at this point.

If m_1, m_2, m_3 denote the middle points of the segments Aa_1, Bb_1, Cc_1, and N the centre of the nine-point circle, it follows from the above that

$Om_1=R-\frac{1}{2}Aa_1$ — for the circle (Aa_1) obviously touches the circle (ABC)

$$Nm_1=\pm\left(\frac{1}{2}Aa_1-\frac{R}{2}\right),$$

or $Om_1\pm Nm_1=\frac{R}{2}=Om_2\pm Nm_2=Om_3\pm Nm_3.$

The + or — sign being taken, according as the triangle is acute- or obtuse-angled.

(If A is a right angle, R will be equal to Aa_1; the point a_1 being, in this case, coincident with O.)

The above relations shew that m_1, m_2, m_3 are three points on a conic whose foci are O and N, and whose transverse axis $=\frac{R}{2}$; this curve being, in fact, the locus of the centres of the conics which touch α, β, γ and pass through O.

Its equation is

$$\sqrt{a\cos A\,(-a\alpha+b\beta+c\gamma)}+\sqrt{b\cos B\,(a\alpha-b\beta+c\gamma)}$$
$$+\sqrt{c\cos C\,(a\alpha+b\beta-c\gamma)}=0\,;$$

For this represents the locus of the centre of the conic

$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0$, when $\sqrt{l\cos A}+\sqrt{m\cos B}+\sqrt{n\cos C}=0$;

If we take the triangle $A_1B_1C_1$ (see explanation of the Notation) as the triangle of reference, the above equation becomes

$$\sqrt{a^2\cos A\,.\,\alpha^1}+\sqrt{b^2\cos B\,.\,\beta^1}+\sqrt{c^2\cos C\,.\,\gamma^1}=0\,;$$

hence the conic in question (which is an ellipse or hyperbola according as the triangle ABC is acute- or obtuse-angled) has for its auxiliary circle the nine-point circle of the triangle $A_1B_1C_1$. See Sect. XVII.

5⁰. The circumscribing and nine-point circles have for a common tangent the directrix of each of the two parabolas which can be drawn through O to touch α, β, γ.

Or, since the directrix of any parabola touching α, β, γ passes through the polar centre P, this will amount to saying that the two common tangents to the circles in question through P—their external centre of similitude—are the directrices of the two parabolas which touch the sides of the triangle, and pass through O.

6⁰. The circles in question will also both pass through the centres of the rectangular hyperbolas (two in this case) which can be drawn through O to touch the sides.

If we compare this result with that obtained in Sect. XXV. it appears that these rectangular hyperbolas must touch the line $\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}$; i.e. the axis of perspective of the triangles ABC, $D'E'F'$.

This may be otherwise shewn, as follows.

The equation $\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0$, will represent the two curves in question if

$$\Sigma l^2\cos^2 A-2\Sigma mn\cos B\cos C=0,$$

and $$\Sigma l^2 + 2\Sigma mn \cos A = 0;$$

hence, by subtracting one of these equations from the other, we have

$$(l \sin A + m \sin B + n \sin C)^2 = 0;$$

and $$\therefore\ l \sin A + m \sin B + n \sin C = 0;$$

which is precisely the condition that the curves shall touch the line $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}$.

Hence, also, it follows that there are only *two* rectangular hyperbolas passing through O and touching α, β, γ: their equations being

$$\sqrt{l_1\alpha} + \sqrt{m_1\beta} + \sqrt{n_1\gamma} = 0, \text{ and } \sqrt{l_2\alpha} + \sqrt{m_2\beta} + \sqrt{n_2\gamma} = 0;$$

where $l_1 : m_1$; l_2, m_2, &c., are the ratios obtained from $\Sigma l \sin A = 0$, and

$$\Sigma l^2 + 2\Sigma mn \cos A = 0.$$

7°. The equations to the two external common tangents of the nine-point and circumscribing circles may be deduced thus.

If $S = 0$, represent the latter circle, the equation

$$lS_a + mS_b + nS_c = 0,$$

may be put under the form

$$\Sigma l \sin B \sin C \,.\, S = L \,.\, \Sigma (l \sin B \sin C \cos A\alpha),$$

which breaks up into the line at infinity L, and the line

$$\Sigma l \sin B \sin C \cos A\alpha; \text{ when } \frac{l}{\sin A} + \frac{m}{\sin B} + \frac{n}{\sin C} = 0;$$

the equations of the common tangents in question will, therefore, be $$l_1 \cot A \,.\, \alpha + m_1 \cot B \,.\, \beta + n_1 \cot C \,.\, \gamma = 0,$$

$$l_2 \cot A \,.\, \alpha + m_2 \cot B \,.\, \beta + n_2 \cot C \,.\, \gamma = 0;$$

where $l_1 : m_1 : n_1$; $l_2 : m_2 : n_2$ are the ratios obtained from

$$\Sigma \frac{l}{\sin A} = 0, \text{ and } \Sigma \sqrt{l \cos A} = 0.$$

XXVII.

The theorem of the last Section, combined with that of Sect. XIX., will enable us to find the condition that the pedal circle of (ξ, η, ζ) shall touch the nine-point circle. For, if we refer to XIX., it will be seen that the pedal of (ξ, η, ζ), the nine-point circle, and the director of the conic

$$\Sigma \sqrt{\begin{vmatrix} \eta & \zeta \\ \cos B & \cos C \end{vmatrix} \quad \begin{vmatrix} \eta \cos C \\ \zeta \cos B \end{vmatrix} \xi \cos A \, . \, a} = 0,$$

are co-axal;

hence the required condition is

$$\cos A \sqrt{\xi(\eta \cos B - \zeta \cos C)(\eta \cos C - \zeta \cos B)}$$

$$+ \cos B \sqrt{\eta(\zeta \cos C - \xi \cos A)(\zeta \cos A - \xi \cos C)}$$

$$+ \cos C \sqrt{\zeta(\xi \cos A - \eta \cos B)(\xi \cos B - \eta \cos A)} = 0;$$

i.e. the locus of a point (ξ, η, ζ) such that the circle passing through the feet of the perpendiculars let fall from it upon the sides of the triangle of reference touches the nine-point circle, is a curve of the sixth order, which passes through

1°. The angular points of the triangle; and the points a_1, b_1, c_1 where the radii AO, BO, CO meet the opposite sides BC, CA, AB. (See 4°. of the preceding Section.)

2°. The centres of the four circles which touch the sides. For the above equation is satisfied by $\frac{\xi}{1} = \frac{\eta}{\pm 1} = \frac{\zeta}{\pm 1}$; i.e. the nine-point circle touches the inscribed and escribed circles; which is Feuerbach's theorem.

If
$$U = \xi(\eta\cos B - \zeta\cos C)\left(\frac{\eta}{\cos B} - \frac{\zeta}{\cos C}\right),$$
$$V = \eta(\zeta\cos C - \xi\cos A)\left(\frac{\zeta}{\cos C} - \frac{\xi}{\cos A}\right),$$
$$W = \zeta(\xi\cos A - \eta\cos B)\left(\frac{\xi}{\cos A} - \frac{\eta}{\cos B}\right).$$

The equation of the curve can be put under the form

$$\sqrt{U\cos A} + \sqrt{V\cos B} + \sqrt{W\cos C} = 0;$$

which shews that it touches each of the nine right lines given by $U = 0$, $V = 0$, $W = 0$; the equations of the cubics of contact being, respectively,

$$V\cos B - W\cos C = 0,$$
$$W\cos C - U\cos A = 0,$$
$$U\cos A - V\cos B = 0.$$

The curve also evidently touches each of the cubics

$$2U\cos A + 2V\cos B - W\cos C = 0,$$
$$2V\cos A - V\cos B + 2W\cos C = 0,$$
$$-U\cos A + 2V\cos B + 2W\cos C = 0.$$

XXVIII.

If x', y' be the extremities of any diameter $x'Oy'$, (see Fig. 14,) of the circle (ABC); the intersections of the circles which bisect the sides of the consecutive triangles $Ax'B$, $x'BC$, BCy', $Cy'A$, $y'Ax'$, will form five points lying on the circumference of the nine-point circle of the triangle ABC.

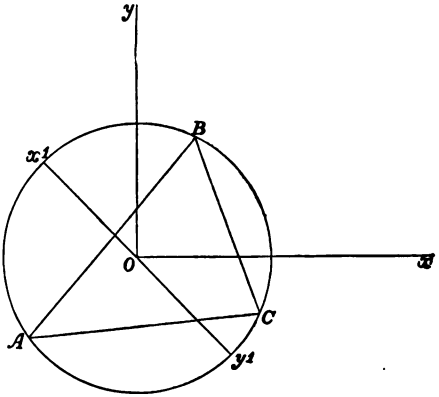

Fig. 14.

This may be deduced from the following:—

Given five points on a circle of radius R; the centres of the five equilateral hyperbolas which pass through them, taken four and four together, will lie on the circumference of another circle, whose radius $= \frac{1}{2} \cdot R$.

If we take any two diameters of the given circles at right angles to each as axes of reference, the equation of the equilateral hyperbola which passes through the four points whose angular ordinates are $\alpha, \beta, \gamma, \delta$ can be put under the form

$$\left\{ x\cos\frac{1}{2}(\alpha+\beta) + y\sin\frac{1}{2}(\alpha+\beta) - R\cos\frac{1}{2}(\alpha-\beta \right\} \times$$

$$\left\{ x\cos\frac{1}{2}(\gamma+\delta) + y\sin\frac{1}{2}(\gamma+\delta) - R\cos\frac{1}{2}(\gamma-\delta) \right\} = \lambda(x^2+y^2-R^2),$$

where

$$2\lambda = \cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\gamma+\delta) + \sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\gamma+\delta)\,;$$

but this evidently reduces to

$$\cos\frac{1}{2}(\alpha+\beta+\gamma+\delta)(x^2-y^2)+2\sin\frac{1}{2}(\alpha+\beta+\gamma+\delta)\,xy$$

$$-2R\left\{\cos\frac{1}{2}(\gamma+\delta)\cos\frac{1}{2}(\alpha-\beta)+\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\gamma-\delta)\right\}x$$

$$-2R\left\{\sin\frac{1}{2}(\gamma+\delta)\cos\frac{1}{2}(\alpha-\beta)+\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\gamma-\delta)\right\}y+F=0,$$

F being the constant term.

From this, the co-ordinates of the centre are found to be

$$x=\frac{1}{2}R(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta),$$

$$y=\frac{1}{2}R(\sin\alpha+\sin\beta+\sin\gamma+\sin\delta):$$

If, then, the angular ordinates of the five given points be ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5, the co-ordinates of the centres of the corresponding equilateral hyperbolas will be

$$\left.\begin{aligned}x&=\frac{1}{2}R\Sigma\cos\phi-\frac{1}{2}R\cos\phi_1\\ y&=\frac{1}{2}R\Sigma\sin\phi-\frac{1}{2}R\sin\phi_1\end{aligned}\right\}\quad\left.\begin{aligned}x&=\frac{1}{2}R\Sigma\cos\phi-\frac{1}{2}R\cos\phi_2\\ y&=\frac{1}{2}R\Sigma\cos\phi-\frac{1}{2}R\sin\phi_2\end{aligned}\right\};$$

&c.,

whence it is easily seen that the five centres lie on the circle whose equation is

$$\left(x-\frac{1}{2}R\Sigma\cos\phi\right)^2+\left(y-\frac{1}{2}R\Sigma\sin\phi\right)^2=\frac{1}{4}R^2.$$

If A, B, C, X, Y denote any five points on a circle, it follows from this that the consecutive intersection of the nine-point circles of the triangles ABC, BCX, CXY, XYA, YAB, lie on another circle of one-half the radius of the first.

Hence, if two of these points (ϕ_4) and (ϕ_5), for instance, are the extremities of a diameter, it is easily seen that the above equation becomes

$$\{x - \frac{R}{2}(\cos\phi_1 + \cos\phi_2 + \cos\phi_3)\}^2 + \{y - \frac{R}{2}(\sin\phi_1 + \sin\phi_2 + \sin\phi_3)\}^2 = \frac{1}{4}R^2:$$

i.e. it represents, in this case, the circle which bisects the sides of the triangle formed by the other three points.

Hence the theorem enunciated at the beginning of the Section.

XXIX.

If four points be taken on the circumference of a given circle, the centres of the nine-point circles of the four triangles that can be formed from them will lie on another circle, whose radius is one-half that of the first.

If ϕ_1, ϕ_2, ϕ_3, ϕ_4 be the angular ordinates of the four points, the co-ordinates of the centres of the nine-point circles of the corresponding triangles are easily found to be

$$\left.\begin{aligned} x &= \frac{R}{2}(\cos\phi_1 + \cos\phi_2 + \cos\phi_3) \\ y &= \frac{R}{2}(\sin\phi_1 + \sin\phi_2 + \sin\phi_3) \end{aligned}\right\}; \quad \left.\begin{aligned} x &= \frac{R}{2}(\cos\phi_2 + \cos\phi_3 + \cos\phi_4) \\ y &= \frac{R}{2}(\sin\phi_2 + \sin\phi_3 + \sin\phi_4) \end{aligned}\right\}.$$

&c.;

hence they all lie on the circle

$$\{x - \frac{R}{2}(\cos\phi_1 + \cos\phi_2 + \cos\phi_3 + \cos\phi_4)\}^2 + \{y - \frac{R}{2}(\sin\phi_1 + \sin\phi^2 + \sin\phi_3 + \sin\phi_4)\}^2 = \frac{1}{4}R^2; \qquad (1)$$

i.e. on a circle whose radius $= \frac{1}{2}R$, and whose centre coincides with the centre of the equilateral hyperbola which passes through $\phi_1, \phi_2, \phi_3, \phi_4$.

Consequences.

If on the circle (ABC) we take two points, x_1, x_2, situated symmetrically with respect to one of the vertices—A, say; and such that arc $Ax_1 = Ax_2 =$ one-sixth of the whole circumference; then the nine-point circle will bisect the distance between the point of intersection of the perpendiculars and the centre of the circumscribed circle in the case of each of the four triangles that can be formed from the quadrangle B, C, x_1, x_2.

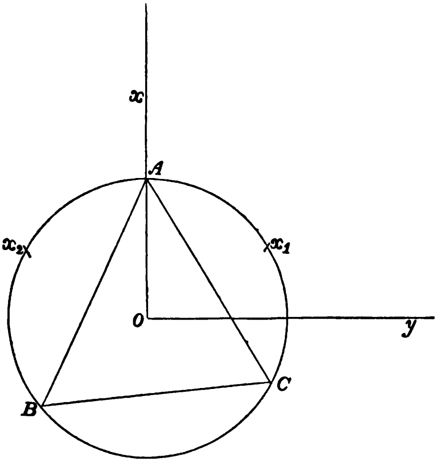

Fig. 15.

For if OA—see Fig. (15)—be taken as the axis of x, equation (1) becomes—since $\phi_1 = -\phi_2 = 60^0$.—

$$\{x - \frac{R}{2}(1 + \cos\phi_3 + \cos\phi_4)\}^2 + \{y - \frac{R}{2}(\sin\phi_3 + \sin\phi_4)\}^2 = \frac{1}{4} \cdot R^2;$$

which is precisely that of the nine-point circle of the triangle ABC; 0, ϕ_3, ϕ_4 being the angular ordinates of the vertices A, B, C.

The above theorem is, of course, applicable to each of the triangles PBC, PCA, PAB, OO_1O_2, OO_2O_3, OO_3O_1, $O_1O_2O_3$.—See Sect. XXII.

It may be also observed that the centre of the nine-point circle coincides with the centre of the equilateral hyperbola which circumscribes the quadrangle B, C, x_1, x_2.

XXX.

If A, B, C, D be any four points on a circle of radius R; the centroids of the triangles ABC, BCD, CDA, DAB will lie on another circle of radius $\frac{1}{3}R$.

For if ϕ_1, ϕ_2, ϕ_3, ϕ_4 be the angular ordinates of the four points in question, the co-ordinates of the centroids of the triangles ABC, &c., will be

$$\left.\begin{aligned} x &= \frac{R}{3}(\cos\phi_1+\cos\phi_2+\cos\phi_3) \\ y &= \frac{R}{3}(\sin\phi_1+\sin\phi_2+\sin\phi_3) \end{aligned}\right\} ; \quad \left.\begin{aligned} x &= \frac{R}{3}(\cos\phi_2+\cos\phi_3+\cos\phi_4) \\ y &= \frac{R}{3}(\sin\phi_2+\sin\phi_3+\sin\phi_4) \end{aligned}\right\} ;$$

hence they, evidently, all lie on the circle

$$\left(x-\frac{R}{3}\Sigma\cos\phi\right)^2+\left(y-\frac{R}{3}\Sigma\sin\phi\right)^2=\frac{1}{9}\,.\,R^2.$$

If this circle be called the centroid-circle of the quadrangle, it may be shewn in the same way that the centroid circles of the five quadrangles that can be formed from five concyclic points will also have their centres on the circumference of another circle of one third the radius of the first.

Mr. Townsend remarks—(see "Educational Times," vol. v. p. 92)—that the first of these theorems may be extended as follows:—"If A, B, C, D, E, F, &c., be the position of any number (n)

of equal masses distributed in space, G that of their centre of gravity, and A', B', C', D', E', F', &c., those of the centres of gravity of their n different groups of $(n-1)$; then always the two systems of n points A, B, C, D, E, F, &c., and A', B', C', D', E', F', &c., are similar, oppositely placed with respect to each other, have G for their centre of similitude, and $(n-1):1$ for their ratio of similitude. The truth of this is evident, for the several lines AA', BB', CC', DD', EE', FF', &c., all connect through G, and are then divided internally in the common ratio of $(n-1):1$."

XXXI.

If F, F' denote any two inverse points (ξ, η, ζ); $\left(\frac{1}{\xi}, \frac{1}{\eta}, \frac{1}{\zeta}\right)$; and x, x'; y, y'; z, z' three pairs of points on the sides BC, CA, AB such that the angles between the directions xF, BC; yF, CA; zF, AB each $=$ given angle θ; and likewise those between $x'F'$, BC; $y'F'$, CA; $z'F'$, AB each $=$ supplement of θ: then the six points in question x, x', y, y', z, z' will lie on the same circle.

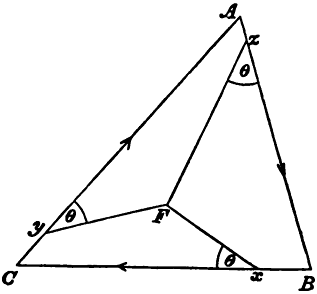

Fig. 16.

If the co-ordinates of F be ξ, η, ζ, those of x—see Fig. (16)—will be 0, $\eta + \xi \sin(\theta + C) \operatorname{cosec} \theta$, $\zeta + \xi \sin(\theta - B) \operatorname{cosec} \theta$:

hence, those of x' will be

$$0, \frac{1}{\eta} + \frac{1}{\xi} \sin(180^\circ - \theta + C) \operatorname{cosec} \theta, \quad \frac{1}{\zeta} + \frac{1}{\xi} \sin(180^\circ - \theta - B) \operatorname{cosec} \theta,$$

or

$$0, \frac{1}{\eta} + \frac{1}{\xi} \sin(\theta - C) \operatorname{cosec} \theta, \qquad \frac{1}{\zeta} + \frac{1}{\xi} \sin(\theta + B) \operatorname{cosec} \theta,$$

with similar expressions for those of y, y', z, z'.

Whence it follows that these six points lie on the conic whose equation is

$$\Sigma \,.\, \xi \{\eta + \zeta \sin(\theta - A) \operatorname{cosec} \theta\} \{\eta \sin(\theta + A) \operatorname{cosec} \theta + \zeta\} \alpha^2$$
$$- \Sigma \{\eta (\xi \sin(\theta + C) \operatorname{cosec} \theta + \eta)(\xi + \zeta \sin(\theta + B) \operatorname{cosec} \theta)$$
$$+ \zeta (\xi + \eta \sin(\theta - C) \operatorname{cosec} \theta)(\xi \sin(\theta - B) \operatorname{cosec} \theta + \zeta)\} \beta\gamma = 0\,;$$

for if $\alpha = 0$, this equation breaks up into the two factors

$$\{\zeta + \xi \sin(\theta - B) \operatorname{cosec} \theta\} \beta - \{\xi \sin(\theta + C) \operatorname{cosec} \theta + \eta\} \gamma,$$

and

$$\eta \{\zeta \sin(\theta + B) \operatorname{cosec} \theta + \xi\} \beta - \zeta \{\xi + \eta \sin(\theta - C) \operatorname{cosec} \theta\} \gamma,$$

i.e. it passes through x, x'; and similarly through y, y'; z, z'.

Again, since the conditions that the conic $\Sigma u\alpha^2 + 2\Sigma u'\beta\gamma$ shall be a circle, are

$$wb^2 + vc^2 - 2u'bc = uc^2 + wa^2 - 2v'ca = va^2 + ub^2 - 2w'ab,$$

we have in the case before us,

$$wb^2 + vc^2 - 2u'bc = \{(\eta + \xi \sin(\theta + C) \operatorname{cosec} \theta) b + (\zeta + \xi \sin(\theta - B) \operatorname{cosec} \theta) c\}$$
$$\times \{\zeta (\xi + \eta \sin(\theta - C) \operatorname{cosec} \theta)\, b + \eta (\xi + \zeta \sin(\theta + B) \operatorname{cosec} \theta) c\}$$
$$= (a\xi + b\eta + c\zeta)(a\eta\zeta + b\zeta\xi + c\xi\eta).$$
$$= uc^2 + wa^2 - 2v'ca = va^2 + ub^2 - 2w'ab.$$

Or the conic is a circle.

If the equation to the above circle, which we may call the Oblique Pedal of (ξ, η, ζ), be written, for shortness, $P_\theta(\xi, \eta, \zeta) = 0$; and if $\xi = \eta = \zeta$, we have

$$P_\theta(1, 1, 1) = \Sigma(1 + \cos A)\alpha^2 - 2\Sigma(1 + \cos B)(1 + \cos C)\beta\gamma$$
$$- \cot^2\theta(\sin A\alpha + \sin B\beta + \sin C\gamma)^2 = 0;$$

which represents a circle concentric with the inscribed circle, and cutting the sides at an angle of $90^0 - \theta$.

In the same way

$$P_\theta(-1, 1, 1) = E_a - \cot^2\theta(\sin A\alpha + \sin B\beta + \sin C\gamma)^2 = 0$$

represents the circle concentric with one of the escribed circles E_a, which cuts the sides at the same angle $90^0 - \theta$.

Hence if $I = 0$ denote the inscribed circle, we have

$$P_\theta(1, 1, 1) - P_\theta(-1, 1, 1) = I - E_a;$$

i.e. the oblique pedals $P_\theta(1, 1, 1)$, $P_\theta(-1, 1, 1)$ intersect on the circle which has for a diameter the line joining the centres of perspective of I and E_a.

Again, since

$$P_\theta(\xi, \eta, \zeta) = \Sigma \,.\, \xi(\eta + \zeta\sin(\theta - A)\operatorname{cosec}\theta)(\eta\sin(\theta + A)\operatorname{cosec}\theta + \zeta)\alpha^2$$
$$- \Sigma \,.\, \{\eta(\xi\sin(\theta + C)\operatorname{cosec}\theta + \eta)(\xi + \zeta\sin(\theta + B)\operatorname{cosec}\theta)$$
$$+ \zeta(\xi + \eta\sin(\theta - C)\operatorname{cosec}\theta)(\xi\sin(\theta - B)\operatorname{cosec}\theta + \zeta)\}\beta\gamma,$$

we have, if

$$\Sigma\xi(\eta + \zeta\sin(\theta - A)\operatorname{cosec}\theta)(\eta\sin(\theta + A)\operatorname{cosec}\theta + \zeta)\sin B\sin C\alpha$$
$$\times \Sigma\sin A\alpha = P_1;$$

$$P_1 - \sin A\sin B\sin C \,.\, P_\theta(\xi, \eta, \zeta)$$
$$= \Sigma \,.\sin A\{\sin^2 B\zeta(\xi + \eta\sin\overline{\theta - C}\operatorname{cosec}\theta)(\xi\sin\overline{\theta + C}\operatorname{cosec}\theta + \eta)$$
$$+ \sin^2 C\eta(\zeta + \xi\sin\overline{\theta - B}\operatorname{cosec}\theta)(\zeta\sin\overline{\theta + B}\operatorname{cosec}\theta + \xi)$$
$$+ \sin B\sin C\eta(\xi\sin\overline{\theta + C}\operatorname{cosec}\theta + \eta)(\xi + \zeta\sin\overline{\theta + B}\operatorname{cosec}\theta)$$
$$+ \sin B\sin C\zeta(\xi + \eta\sin\overline{\theta - C}\operatorname{cosec}\theta)(\xi\sin\overline{\theta - B}\operatorname{cosec}\theta + \zeta)\}\beta\gamma.$$

$$= \Sigma \left\{ \sin B \left(\eta + \xi \sin \overline{\theta + C} \operatorname{cosec} \theta\right) + \sin C \left(\zeta + \xi \sin \overline{\theta - B} \operatorname{cosec} \theta\right) \right\}$$

$$\times \left\{ \sin B \zeta \left(\xi + r \sin \overline{\theta - C} \operatorname{cosec} \theta\right) + \sin C \eta \left(\xi + \zeta \sin \overline{\theta + B} \operatorname{cosec} \theta\right) \right\} \sin A \beta\gamma.$$

$$= \Sigma . (\sin A\xi + \sin B\eta + \sin C\zeta)(\sin A\eta\zeta + \sin B\zeta\xi + \sin C\xi\eta) \sin A\beta\gamma.$$

∴ The equation to the pedal of (ξ, η, ζ) may be written in the form

$$\Sigma \sin A\alpha . \Sigma \xi \left(\eta + \zeta \sin \overline{\theta - A} \operatorname{cosec} \theta\right) \left(\eta \sin \overline{\theta + A} \operatorname{cosec} \theta + \zeta\right) \sin B \sin C\alpha$$

$$= \Sigma \sin A\xi \ . \ \Sigma \sin A\eta\zeta \ . \ \Sigma \sin A\beta\gamma.$$

If (ξ, η, ζ) lies on the circumscribing circle, we have

$$\Sigma \sin A\eta\zeta = 0,$$

and the pedal breaks up into the line at infinity $\Sigma \sin A\alpha$, and

$$\Sigma \xi \left(\eta + \zeta \sin \overline{\theta - A} \operatorname{cosec} \theta\right) \left(\eta \sin \overline{\theta + A} \operatorname{cosec} \theta + \zeta\right) \frac{\alpha}{\sin A} = 0.$$

Also the equation of the radical axis of $P_\theta (\xi, \eta, \zeta)$ and the circumscribed circle is easily seen to be

$$\Sigma \xi \left(\eta + \zeta \sin \overline{\theta - A} \operatorname{cosec} \theta\right) \left(\eta \sin \overline{\theta + A} \operatorname{cosec} \theta + \zeta\right) \frac{\alpha}{\sin A} = 0,$$

or, what is the same thing,

$$\sin^2 \theta \, . \, \Sigma (\eta + \zeta \cos A) \left(\frac{1}{\eta} + \frac{1}{\zeta} \cos A\right) \frac{\alpha}{\sin A}$$

$$+ \sin \theta \cos \theta \ . \ \Sigma \left(\frac{\eta}{\zeta} - \frac{\zeta}{\eta}\right) \alpha$$

$$- \cos^2 \theta \ . \ \Sigma \sin A\alpha = 0 ;$$

hence, if (ξ, η, ζ) be a *fixed* point, the envelope of the radical axis in question will be given by the equation

$$4 \Sigma \sin A\alpha \ . \ \Sigma (\eta + \zeta \cos A) \left(\frac{1}{\eta} + \frac{1}{\zeta} \cos A\right) \frac{\alpha}{\sin A}$$

$$+ \left\{ \left(\frac{\eta}{\zeta} - \frac{\zeta}{\eta}\right) \alpha + \left(\frac{\zeta}{\xi} - \frac{\xi}{\zeta}\right) \beta + \left(\frac{\xi}{\eta} - \frac{\eta}{\xi}\right) \gamma \right\}^2 = 0 ;$$

which represents a parabola whose axis is parallel to the line $\Sigma\left(\frac{\eta}{\zeta}-\frac{\zeta}{\eta}\right)\alpha = 0$; i.e. to the line which joins (ξ, η, ζ) to $\left(\frac{1}{\xi}, \frac{1}{\eta}, \frac{1}{\zeta}\right)$.

Similarly, if (ξ, η, ζ) be supposed to be a fixed point, it may be shewn that the envelope of the circle $P_\theta\,(\xi, \eta, \zeta)$ is a curve of the fourth order.

If the triangles ABC, xyz—see Fig. (16)—are in perspective, the locus of F will be the cubic

$$(\eta\sin\overline{\theta+A}\operatorname{cosec}\theta+\zeta)(\zeta\sin\overline{\theta+B}\operatorname{cosec}\theta+\xi)(\xi\sin\overline{\theta+C}\operatorname{cosec}\theta+\eta)$$

$$=(\eta+\zeta\sin\overline{\theta-A}\operatorname{cosec}\theta)(\zeta+\xi\sin\overline{\theta-B}\operatorname{cosec}\theta)(\xi+\eta\sin\overline{\theta-C}\operatorname{cosec}\theta).$$

Since this equation remains unchanged when ξ, η, ζ are replaced by $\frac{1}{\xi}, \frac{1}{\eta}, \frac{1}{\zeta}$, and θ by $180^0-\theta$, it follows that, if the triangle of reference is in perspective with xyz, it will also be in perspective with the triangle $x'y'z'$.

If $\theta = 90^0$, the equation to the cubic in question becomes

$$(\eta\cos A+\zeta)(\zeta\cos B+\xi)(\xi\cos C+\eta)=(\eta+\zeta\cos A)(\zeta+\xi\cos B)(\xi+\eta\cos C),$$

i.e. the locus of a point F such that the triangle xyz—formed by the feet of the perpendiculars drawn from it on the sides BC, CA, AB—shall be in perspective with the triangle ABC, is a curve of the third order, which passes through

1°. The angles of the triangle ABC.

2°. The centres of the four circles which touch the sides.

3°. The point of intersection of the three perpendiculars.

4°. The centre O, of the circumscribing circle.

5°. The points where the radii AO, BO, CO produced meet this circle again.

Moreover the cubic is central, having O for its centre; and since its equation remains unchanged when ξ, η, ζ are replaced by $\frac{1}{\xi}$, $\frac{1}{\eta}$, $\frac{1}{\zeta}$, it is also its own inverse.

APPENDIX.

Note on the Theorem of Sect. XXVIII.

If the middle points of the sides A_1, B_1, C_1 and the feet of the perpendiculars D, E, F of any triangle ABC be divided into six groups of five points, it appears from Sect. XXVIII. that to each group there will correspond a certain circle; viz. the circle which passes through the centres of the five equilateral hyperbolas drawn through the points comprised in the group—taken four and four together. This being so, it may be easily shewn that the centres of these six circles will all lie on the circumference of another circle of radius $\frac{1}{4} . R$; R being the radius of the circle (ABC).

For, if any two rectangular diameters of the circle $(A_1B_1C_1DEF)$ be taken as axes of reference, and $\phi_1, \phi_2 \dots\dots \phi_6$ be the angular ordinates of A_1, B_1, C_1, D, E, F, the equations of the six circles in question will be

$$\{x - \frac{R}{4}(\cos\phi_1 + \cos\phi_2 + \cos\phi_3 + \cos\phi_4 + \cos\phi_5)\}^2$$

$$+ \{y - \frac{R}{4}(\sin\phi_1 + \sin\phi_2 + \sin\phi_3 + \sin\phi_4 + \sin\phi_5)\}^2 = \frac{1}{16} . R^2$$

$$\{x - \frac{R}{4}(\cos\phi_2 + \cos\phi_3 + \cos\phi_4 + \cos\phi_5 + \cos\phi_6)\}^2$$

$$+ \{y - \frac{R}{4}(\sin\phi_2 + \sin\phi_3 + \sin\phi_4 + \sin\phi_5 + \sin\phi_6)\}^2 = \frac{1}{16} . R^2.$$

&c.

The co-ordinates of their centres are, therefore,

$$\left.\begin{array}{l} x = \frac{R}{4}\,\Sigma\cos\phi - \frac{R}{4}\cos\phi_6 \\ y = \frac{R}{4}\,\Sigma\sin\phi - \frac{R}{4}\sin\phi_6 \end{array}\right\}; \quad \left.\begin{array}{l} x = \frac{R}{4}\,\Sigma\cos\phi - \frac{R}{4}\cos\phi_1 \\ y = \frac{R}{4}\,\Sigma\sin\phi - \frac{R}{4}\sin\phi_1 \end{array}\right\};$$

&c.

which, evidently, will lie on the circle

$$\left(x - \frac{R}{4}\,\Sigma\cos\phi\right)^2 + \left(y - \frac{R}{4}\,\Sigma\sin\phi\right)^2 = \frac{1}{16}R^2$$

Q.E.D.

The result arrived at in Sect. XXVIII. leads also to the following curious extension of a known theorem:—

If we are given any five right lines, M. Auguste Miquel has shewn (see Catalan's *Théorèmes et Problèmes de Géométrie Elémentaire*) that the foci of the five parabolas which touch them, taken four and four together, lie on a circle;

Hence, if five equilateral hyperbolas be made to pass through these foci, taken four together, we learn from the result before us that their centres will lie on a second circle of one-half the radius of the first or five-focus circle.

I may observe that the theorem under consideration may be derived from the following:—

Given five points A, B, C, D, E on any conic section K; then the consecutive intersections of the five conics, each similar and similarly placed to K, which bisect the sides of the triangles ABC, BCD, CDE, DEA, EAB, lie on a sixth similar and similarly placed conic, whose axes = one-half those of K.

For example, if we take K to be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

and A, B, C, D, E to be the points $(a\cos\phi_1, b\sin\phi_1)$; $(a\cos\phi_2,$

$b \sin \phi_2$); ($a \cos \phi_3$, $b \sin \phi_3$); ($a \cos \phi_4$, $b \sin \phi_4$); ($a \cos \phi_5$, $b \sin \phi_5$;) it is easily seen that the equations of the two ellipses, similar and similarly placed to *K*, which bisect the sides of the triangles *ABC*, *BCD*, for instance, are

$$\frac{\{x - \frac{a}{2}(\cos\phi_1 + \cos\phi_2 + \cos\phi_3)\}^2}{a^2} + \frac{\{y - \frac{b}{2}(\sin\phi_1 + \sin\phi_2 + \sin\phi_3)\}^2}{b^2} = \frac{1}{4},$$

$$\frac{\{x - \frac{a}{2}(\cos\phi_2 + \cos\phi_3 + \cos\phi_4)\}^2}{a^2} + \frac{\{y - \frac{b}{2}(\sin\phi_2 + \sin\phi_3 + \sin\phi_4)\}^2}{b^2} = \frac{1}{4};$$

and, consequently, that they pass through the common point

$$\left.\begin{aligned} x &= \frac{a}{2}(\cos\phi_1 + \cos\phi_2 + \cos\phi_3 + \cos\phi_4) \\ y &= \frac{b}{2}(\sin\phi_1 + \sin\phi_2 + \sin\phi_3 + \sin\phi_4) \end{aligned}\right\};$$

Hence, the co-ordinates of the five consecutive points of intersection of the five ellipses before us are

$$\left.\begin{aligned} x &= \frac{a}{2}\Sigma\cos\phi - \frac{a}{2}\cos\phi_5 \\ y &= \frac{b}{2}\Sigma\sin\phi - \frac{b}{2}\sin\phi_5 \end{aligned}\right\}; \quad \left.\begin{aligned} x &= \frac{a}{2}\Sigma\cos\phi - \frac{a}{2}\cos\phi_1 \\ y &= \frac{b}{2}\Sigma\sin\phi - \frac{b}{2}\sin\phi_1 \end{aligned}\right\};$$

&c.

which will all lie on the similar and similarly placed ellipse

$$\frac{\{x - \frac{a}{2}\Sigma\cos\phi\}^2}{a^2} + \frac{\{y - \frac{b}{2}\Sigma\sin\phi\}^2}{b^2} = \frac{1}{4}$$

Note on Sect. XXII.

THE property of the nine-point circle given on page 41—viz. that it is the polar circle of the triangle whose vertices are (B_1C_1, EF), (C_1A_1, FD), (A_1B_1, DE)—was first noticed by M. Schroeter of Breslau. See the *Nouvelles Annales de Mathématiques* for October, 1864, p. 442.

The proof in the text is, however, the Author's.

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